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Almost maximal, e-maximal and e-simple fuzzy submodules

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Abstract

In this paper, the concepts of almost maximal submodules are fuzzified and studied. Also, three new concepts are introduced, which are: e-simple submodules, Ee-simple modules and e-maximal submodules. Then these concepts are fuzzified and studied.

Keywords: Almost maximal fuzzy submodules, e-maximal fuzzy submodules, Ee-simple fuzzy modules and Ee-simple fuzzy module.

1. Introduction

Let R be a commutative ring with unity and V be an R-module. Recall that a proper submodule U of V is called a maximal submodule if V has no properly. Many generalizations of maximal submodules were introduced such that semimaximal submodules, weakly maximal submodules and 2-maximal submodules [3, 4], Inaam and Riadh in [6] gave another type of maximal submodules namely "almost maximal submodules"), where proper submodule U of V is called almost maximal if V has no proper essential submodule W containing U properly; that is $U < W$and W is essential submodule in V, then $U = W$.

In fact, a submodule W of V is called an essential($W \leq_e V$). If whenever $K \leq V, W \cap K = (0)$. Then $K = (0)$. These concepts maximal submodules, semimaximal submodules, weakly maximal submodules and 2- maximal submodules had been fuzzified. Moreover, the concept of essential submodules had been fuzzified and studied in [11]. This lead us to introduce the concept of almost maximal submodules, where a fuzzy submodule H of a fuzzy module X is said to be almost maximal if there is a proper essential fuzzy submodule containing H properly. So that S.3 of this paper

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is devoted to studying almost maximal fuzzy submodules. Besides this, H is well known that a submodule U of V is called simple if U has no proper nonzero submodule, and V is a simple module if V is a simple submodule of V. Also, a proper submodule U of V is maximal if and only if $V/U$is a simple R-module. These observations lead us to introduce three concepts: e-simple, Extremely e-simple module(Ee-simple module and e-maximal submodule, where a nonzero submodule U of V is called e-simple if there is no $W \leq_e$ V such that $W \leq U$, that is (if $W \leq U$ and $W \leq_e V$, then $W = U$. An R-module V is called Extremely e-simple (briefly Ee-simple) if every nonzero submodule U of V is e-simple. A proper submodule U of V is called e-maximal if V/U is Ee-simple R-module. Then in Section 4, these concepts are studied, where many properties and relations are introduced.

In Section 5, the concepts which are given in Section 4 are fuzzified and studies, where many properties transform to fuzzy submodules and to fuzzy modules.

2. Preliminaries

This section contains some definitions and properties of fuzzy subset, and fuzzy modules submodules), which we used in the next sections.

Definition 2.1. *[12] Let S be a non-empty set and I be the closed interval $[0, 1]$ of the real line (real numbers). An F-set A in S (an F-subset of S) is a function from S into I".*

Definition 2.2. *[8] Let A be an F- set in S, for all $t \in [0, 1]$, the set $A_t = \{x \in S, A(x) \geq t\}$ is called a level subset of A.*

Note that, $A(t)$ is a subset of S in the ordinary sense. Moreover, $A_* = A_{A(0)} = \{x \in S, A(x) = A(0)\}$.

Definition 2.3. *[8] Let M be an $R-$module. A fuzzy set X of M is called F-module of an $R-$module M if:*

(i) $X(x - y) \geq min\{X(x), X(y), for\ all\ x, y \in M\}$,
(ii) $X(rx) \geq X(x)$, *for all* $x \in M$ *and* $r \in R$,
(iii) $X(0) = 1$.

Definition 2.4. *[7] Let X and A be two F- modules of $R-$module M. A is called an F-submodule of X if $A \leq X$*

Proposition 2.5. *[7] Let A be an F-set of an $R-$module M.Then the level subset $A_t\, t \in [0, 1]$ is a submodule of M if and only if A is an F-submodule of X where X is a F module of an $R-$module M.*

Definition 2.6. *[9] A fuzzy module X is called simple if X has no nontrivial fuzzy submodules.*

In other words; X is simple if whenever $A \leq X$, either $A = X$ *or* $A = 0_1$.
Moreover, let $A \leq X$, A is a fuzzy simple submodule of X if A is a fuzzy simple module.

Definition 2.7. *[9] A fuzzy module X is called semisimple if X is sum of simple fuzzy submodule of X.*

Definition 2.8. *[10] Let A be a proper fuzzy submodule of fuzzy module X , A is called fuzzy maximal if $\frac{X}{A}$ is a fuzzy simple module. Equivalently A is a maximal fuzzy submodule if whenever $A < B$ and $B \leq X$, then $B \leq X''$*

Definition 2.9. *[10] Let A be a fuzzy submodule of fuzzy module X, then A is called semimaximal if and only if $\frac{X}{A}$ is a fuzzy semisimple module.*

3. Almost maximal fuzzy submodules

Definition 3.1. *A is called almost maximal if there is no proper essential fuzzy submodule B containing A properly, where* $A, B < X$ *i.e (if* $A < B \leq_e X$*, then* $B = X$ *).*

3.1. Remarks and Examples

(i) It is clear that every maximal fuzzy submodule is almost maximal fuzzy submodule, but the converse may not be true as the following example shows:
Let $M = Z_{42}$ *as* $Z - module$, and let $X : M \to [0,1]$ defined by $X(x) = 1,\ \forall x \in Z_{42}$
Define $A : M \to [0,1]$, by $A(x) = \begin{cases} 1 & if\ x \in <\bar{6}>, \\ 0 & \text{otherwise} \end{cases}$.
The only fuzzy submodules which containing A properly are $B : M \to [0,1]$, $C : M \to [0,1]$, with the forms

$$B(x) = \begin{cases} 1 & \text{if } x \in <\bar{2}>, \\ \alpha \in [0,1) & \text{otherwise} \end{cases}, \qquad C(x) = \begin{cases} 1 & \text{if } x \in <\bar{3}>, \\ \beta \in [0,1) & \text{otherwise} \end{cases}$$

However $B_* =<\bar{2}>$ *and* $C_* = <\bar{3}>$ are not essential in M, so that B and C are not essential in X ,by ([4], Proposition 2.1.4). Thus A is fuzzy almost maximal submodule.

(ii) Let $X : Z_{12} \to [0,1]$ such that $X(x) = 1,\ \forall x \in Z_{12}$
Define $A : Z_{12} \to [0,1]$ *by* $A(x) = \begin{cases} 1 & \text{if } x \in <\bar{6}>, \\ 0 & \text{otherwise} \end{cases}$.
A is not almost maximal fuzzy submodule, since there exists B where $B : Z_{12} \to [0,1]$ defined by, $B(x) = \begin{cases} 1 & x \in <\bar{2}>, \\ 0 & \text{otherwise} \end{cases}$.
As for each , $t > 0$, $B_t =<\bar{2}>\leq_e X_t = Z_{12}$, $B <_e X$[6]. Thus $A < B <_e X$.

(iii) Let X be a uniform fuzzy module (every fuzzy submodule of X is essential in X. Let $A < X$.Then A is maximal fuzzy submodule of X is if and only if A is almost maximal fuzzy submodule of X.

(iv) Let X be a chained fuzzy module $\forall A$, $B \leq X$ *either* $A \leq B$ *or* $B \leq X$) and let $A < X$. A is maximal fuzzy submodule of X is if and only if A is almost maximal fuzzy submodule of X .

Recall that for any fuzzy submodule A and B. If A and B are F-submodules of F- module X such that $A \leq B$, then $A_* \leq B_*$ .However the converse is not true [10].

In this paper we denote the following condition by (*) hold and we shall use it in some results. For any fuzzy submodules A and B of fuzzy module X, $A_* \leq B_* \Leftrightarrow A \leq B$.

Theorem 3.2. *A is an almost maximal fuzzy submodule of X* $\Leftrightarrow A_*$ *is an almost maximal submodule of* X_*. *Provided condition (*).*

Proof .

$\Longrightarrow$ *Assume* $A_* < N \leq X_*$ *and* $N \leq_e X_*$.
Define $B : M \to [0,1]$, *Defined by:* $B(x) = \begin{cases} 1 & \text{if } x \in N, \\ 0 & \text{otherwise} \end{cases}$.
Then, $B_* = N \leq_e X_*$, *so* $A_* < B_* \leq X_*$.
By condition (*), $A < B \leq X$. *But* $B \leq_e X$. *Thus*$B = X$ *and so* $B_* = N = X_*$. *Thus* A_* *is an*

almost maximal submodule of X_*.
$\Longleftarrow$ *Assume* $A < B \leq_e X$. *As* $B \leq_e X$, $B_* \leq_e X_*$*([4] ,Proposition 2.1.4). Also* $A_* < B_* \leq X_*$. *But* A_**is almost maximal submodule in* X_*. *So* $B_* = X_*$ *and by condition (*)* $B = X$. *Therefore A is an almost maximal fuzzy submodule.* □

Remark 3.3.
• *Let* $A < B < X$. *A is almost maximal fuzzy submodule of X, then B is almost maximial fuzzy submodule of X .Provided condition(*).*
Proof . *By Theorem 3.2,* A_**is an almost maximal submodule in* X_* . *But* $A_* < B_*$, *hence* B_* *is almost maximal submodule in* X_* *([6], Remark 1.2(2)). Then, by Theorem 3.2, B is an almost maximal fuzzy submodule of X.* □
• *If A and B are almost maximal F- submodules of F- module X, then* $A \cap B$ *is need not be almost maximal t of X.*
Example :
Let M be the Z-module Z. Define $X : M \to [0,1]$ *by* $X(x) = \begin{cases} 1 & if x \in 2Z \ , \\ \frac{1}{2} & otherwise \end{cases}$. *So* $X_* = 2Z$.
Let $A, B : M \to [0,1]$, *defined by:* $A(x) = \begin{cases} 1 & if x \in 4Z \ , \\ \frac{1}{2} & otherwise \end{cases}$, $B(x) = \begin{cases} 1 & if x \in 6Z \ , \\ \frac{1}{2} & otherwise \end{cases}$.
A and B are maximal fuzzy submodules in X ([10],Remark 2.2.1). Hence A and B are almost maximal fuzzy submodules of X by Remark 3.1. Now

$$A \cap B(x) = \begin{cases} 1 & if x \in 12Z \ , \\ \frac{1}{2} & otherwise \end{cases}.$$

$A \cap B < B$. *But* $B_t = 6Z <_e X_t = 2Z$, $\forall t > \frac{1}{2}$. *And for all* $0 < t \leq \frac{1}{2}, B_t = 2Z = X_t$. *Hence* $B \lneqq_e X$ *[2] and so* $A \cap B$*is not almost maximal.*

Proposition 3.4. *A homomorphic image of almost maximal fuzzy submodule is almost maximat fuzzy submodule. Provided condition (*).*
Proof . *Assume* $f : X \to X'$ *be a fuzzy epimorphism and A is an almost maximal fuzzy submodule. To prove* $f(A)$ *is an almost maximal fuzzy submodule of* X'.
By Theorem 2.3 , A_* *is almost maximal submodule in* X_*, *hence by ([6], Remark 1.2(5))* $f(\ A_*)$ *is an almost maximal submodule in* $(X)'_*$. *But* $f\ (\ A_*) \leq f\ (A)_*$. *It follows that* $f\ (A)_*$ *is an almost maximal submodule in* X'_* *(by ([6], Proposition. 3.1). Thus* $f(A$ *is almost maximal fuzzy submodule of* X' *by Theorem 2.3* □

Proposition 3.5. *Let M be an R-module,* $X : M \to [0,1]$*defined by* $X(x) = 1, \forall x \in M$. *If A is an almost maximal fuzzy submodule. Then* X/A *is semismiple fuzzy module.*
Proof . *Suppose A is an almost maximal fuzzy submodule,* A_* *is an almost maximal submodule in* X_*, *Then* X_*/A_* *semismiple by ([6], Theroem 1.10). But* $(X/A)_* = X_*/A_*$. *by ([10], Proposition 1.3.7). Thus* $(X/A)_*$ *is semismiple and by ([10], Proposition 2.1.11)* X/A *is semismiple.* □

Definition 3.6. *Let A be a F- submodule of F- module X, then A is called semimaximal if and only if* X/A *is a semisimple F- module.*

Corollary 3.7. *Let M be an R-module,* $X : M \to [0,1]$ *defined by* $X(x) = 1, \ \forall x \in M$ If $A < X$ *and A is an almost maximal of X , then A is semimaximal.*
Proof . *By previous proposition,* X/A *is semisimple, hence A is semimaximal.* □

Remark 3.8. *The converse of Coorolary 3.7 is not true in general, for example:*

Example 3.9. *Let M be the Z-module Z .Define* $X : M \to [0, 1]$*, by*

$$X(x) = 1, \forall x \in Z. A(x) = \begin{cases} 1 & \text{if } x \in 6Z \ , \\ 0 & \text{otherwise} \end{cases}$$

$Z/A_* = Z/6Z \cong z_6 \to [0, 1]$*, since* Z_6 *is semisimple Z-module , X/A is semisimple fuzzy module by ([10], Remark 2.1.10) and so A is semimaximal . But* $A < B <_e < X$*, where*

$$B(x) = \begin{cases} 1 & \text{if } x \in 2Z \ , \\ 0 & \text{otherwise} \end{cases}.$$

It is easy show that $B_t = 2Z <_e X_t = 2Z, \forall t > 0$ *and so* $B <_e X$ *by [4].*
Thus A is not almost maximial

Proposition 3.10. *Let X be a fuzzy module. Then the following statements are equivalent:*

(i) $(0)_1$ *is an almost maximal fuzzy submodule.*
(ii) Every fuzzy submodule of X is almost maxima.
(iii) X is semisimple.

Proof .
(i)$\Longrightarrow$ *(iii) Since* $(0)_1$ *is an almost maximal fuzzy submodule, then by Proposition 3.4* $X/(0)_1$ *is semisimple. Thus X is semisimple.*
(iii)$\Longrightarrow$ *(i) By ([10], Proposition 2.1.9) , X is has no proper essential fuzzy submodule. Thus there is no proper essential fuzzy submodule containing* $(0)_1$ *properly. Thus* $(0)_1$ *is an almost maximal fuzzy submodule.*
(ii)$\Longrightarrow$ *(i) It is obvious.*
(i)$\Longrightarrow$ *(ii) Let* $A < X$*. If there is* $B <_e X$ *and* $A \lneqq B$*. Then* $(0)_1 \leq B$*. and so* $(0)_1$ *is not almost maximal fuzzy submodule, which is a contradiction.* □

One might ask whether or not the direct sum of almost maximal fuzzy submodule. The answer is the direct sum of almost maximal fuzzy submodule may be not almost maximal as the following example shows:

Example 3.11.

$$X(x) = \begin{cases} 1 & \text{if } x \in 2Z \ , \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$

Then

$$(X \bigoplus X)(x, y) = \begin{cases} 1 & \text{if } x \in 2Z \bigoplus 2Z \ , \\ \frac{1}{2} & \text{otherwise} \end{cases},$$

Define $A, B : M \to [0, 1]$ *by*

$$A(x) = \begin{cases} 1 & \text{if} x \in 4Z \ , \\ \frac{1}{2} & \text{otherwise} \end{cases}, \quad B(x) = \begin{cases} 1 & \text{if } x \in 6Z \ , \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$

A, B are almost maximal fuzzy submodules in X

$$A \bigoplus B(x,y) = \begin{cases} 1 & if\ x \in 4Z \bigoplus 6Z\ , \\ \frac{1}{2} & otherwise \end{cases} .$$

$A \bigoplus B \gneqq X \bigoplus B \gneqq X \bigoplus X$ *,where*

$$(X \bigoplus B)(x,y) = \begin{cases} 1 & if\ x \in 2Z \bigoplus 6Z\ , \\ \frac{1}{2} & otherwise \end{cases} .$$

But $X \bigoplus B <_e X \bigoplus X$, *since* $\forall\ t > \frac{1}{2}$, $(X \bigoplus B)_t = X_t \bigoplus B_t = 2Z \bigoplus 6Z \leq_e X_t \bigoplus X_t = 2Z \bigoplus 2Z$ *and* $\forall\ t < \frac{1}{2}$, $(X \bigoplus B)_t = X_t \bigoplus B_t = 2Z \bigoplus 2Z \leq_e X_t \bigoplus X_t = 2Z \bigoplus 2Z$.

However with extra assumptions, the direct sum of almost maximal fuzzy submodules can be almost maximal fuzzy submodule , but first the following lemma is needed.

Lemma 3.12. *If N, W are almost maximal submodules of the R-modules U, V then $N \bigoplus W$ is an almost maximal submodule in $U \bigoplus V$. Provided ann U + ann$V = R$.*
Proof . *Assume $N \bigoplus W < Y \leq_e U \bigoplus V$. As ann U + ann$V = R$. Then $Y = C \bigoplus D$ by ([1], Proposition 4.2), for some $C \leq U$ and $D \leq V$. Moreover, $Y = C \bigoplus D \leq_e U \bigoplus V$. implies $C \leq_e U$ and $D \leq_e V$. ([5], Proposition 1.1) It follows that $N < C \leq_e U$ and $W < D \leq_e V$. But N and W are almost maximal submodules in U, V respectively. So that $C = U$ and $D = V$. Thus $Y = C \bigoplus D$ and $N \bigoplus W$ is an almost maximal in $U \bigoplus V$.* □

Theorem 3.13. *Let X and Y be fuzzy modules over R- modules $M_1 and M_2$. If G , H are almost maximal fuzzy submodules of X and Y (respectively), then $G \bigoplus H$ is almost maximal in $X \bigoplus Y$. Provided annX_* + ann $Y_* = R$. Provided condition(*).*
Proof . *Since G, H are almost maximal fuzzy submodules of X and Y (respectively), then $G_*$and H_* are almost maximal submodules in X_* and Y_* (respectively, by Theorem 3.2). By previous Lemma $G_* \bigoplus BH_*$ is an almost maximal submodule in $X_* \bigoplus Y_*$. It follows that $(G+H)_*$ is almost maximal in $(X+Y)_*$, since $(G+H)_* = G_* \bigoplus H_*$ and $(X+Y)_* = X_* \bigoplus Y_*$ [11]. Thus by Theorem 3.2, $A \bigoplus B$ is fuzzy an almost maximal in $X \bigoplus Y$.* □

Next we have the following:

Proposition 3.14. *Let A and B be fuzzy submodules of fuzzy module X ,Y(respectively). If $A \bigoplus B$ is an almost maximal in $X \bigoplus Y$. Then A and B are almost maximal fuzzy submodules of X ,Y (respectively).*
Proof . *Let $P_1,, P_2$ be a natural projection from $A \bigoplus B \to A$, $A \bigoplus B \to B$, respectively, then by Proposition 2.4, A and B are almost maximal of X ,Y respectively.* □

4. e-simple submodule, Ee –simple modules and e-maximal submodules

It is well-known that :A proper submodule N of an R-module M is maximal if and only if M/N is simple R-module.
This motivate us to introduce new concepts e-maximal submodule, Extremely e-simple module and e-simple submodule Basic properties about these concepts are investigated and many relations with other classes of submodule (modules) are given.

Definition 4.1. *Let M be an R-module. $(0) \neq N < M$, then N is called e-simple submodule,if there is no $W <_e M$ and $W < N$. M is called Extremely e-simple(Ee-simple) module if every non-zero proper submodule of M is e-simple. If $N < M$, then N is called e-maximal submodule of M, if $\frac{M}{N}$ is Ee-simple R-module.*

4.1. Basic properties and Examples

(i) *It is clear that every simple submodule is e- simple module and every simple module is Ee- simple module , but not conversely where* $N =< \bar{2} >< Z_{12}$ *(as − Zmodule) is e-simple submodule, but it is not simple. Also* $M = Z_{12}$ *as − Zmodule is Ee-simple module, but M is not simple module..*

(ii) *Every non zero proper submodule of the Z –module Z is not e-simple submodule.*

(iii) *Let* $M= Z_8$ *as − Zmodule ,* $N =< \bar{2} >< Z_8$ *is not e-simple submodule, but* $K =< \bar{4} >$*is e-simple submodule. Thus* Z_8 *is not Ee-simple module. However* $M= = Z_4$ *is Ee-simple module.*

(iv) *Every semisimple module is Ee-simple ,but the converse is not true, see part (iii).*

(v) *If M is uniform or chained module ,* $N < M$*, then N is e-simple submodule iff N is simple submodule*

(vi) *Let M be the Z- module Z,* $N =< 4 >, W =< 12 >$*. N, W are e-maximal submodules, since* $\frac{M}{N} \approx Z_4$ *and* $\frac{M}{W} \approx Z_{12}$ *are Ee-simple modules.*

(vii) *Let* $N < M$*, consider the following:*
(a) N is a maximal submodule
(b) N is almost maximal submodule
(c) N is a semimaximal submodule.
(d) N is e- maximal submodule. Then:
$(a) \Longrightarrow (b) \Longrightarrow (c)$*, but the reverse inclusion may be not hold.*
$(a) \Longrightarrow (b)$ *see [6].*
$(b) \Longrightarrow (c)$ *see [6].*
$(3) \Longrightarrow (d)$ *Since N is a semimaximal submodule, then submodule,* M/N *is semisimple and hence by part (v),* M/N *is Ee-simple. Thus N is e- maximal submodule.*
By ([6], Exam. 2.3) $(b) \not\Rightarrow (a)$*, also* $(c) \not\Rightarrow (b)$ *[6].*
Now consider $N = 4Z < Z$ *as Z-module*
$Z/4Z \approx Z_4$ *is Ee- simpl module, so* $N = 4Z$ *is e-maximal , but N is not semimaximal submodule since* $Z/4Z \approx Z_4$*is not semisimple.*

(viii) *It is clear that if* $N < W < M$*, W is e-simple submodule, then N is e-simple. Hence the intersection (if the intersection*$\neq (0)$ *of any two e- simple submodules is e- simple.*

(ix) *If* $N < W < M$*, W is e- simple submodule in M , then*$\frac{W}{N}$ *is e- simple submodule in*$\frac{M}{N}$*.*
Proof . *Assume there exists* $U/N < M/N$ *and* $U/N < W/N$*.*
Then $U <_e M$ *and* $U < W$ *by ([1], Proposition 1.1.c) and so W is not e- simple submodule which condration.* □

(x) *If M is Ee- simple R-module and* $N < M$*, then* M/N *is Ee-simple R-module .*
Proof . *It follows by part (ix).* □

(xi) *The converse of part (x) may be is not true in general as the following example shows:*
Example: Z as Z- module is not e- simple, but $Z/< 4 >\approx Z_4$*is Ee-simple module.*

(xii) *If* M_1*and* M_2 *are R- modules such that* $M_1 \bigoplus M_2$ *is Ee-simple, then* M_1*and* M_2 *are Ee-simple R- modules.*
Proof . *Since* $(M_1 \bigoplus M_2)/M_1 \approx M_2$ *and* $(M_1 \bigoplus M_2)/M_2 \approx M_1$ *and* $M_1 \bigoplus M_2$ *is Ee-simple module . Hence by(x),* M_1*and* M_2 *are Ee-simple R- modules.* □

(xiii) *The converse of (xii) may by false, see following example :*
Example: Let $M_1 = Z_{12}$ *as Z-module,* $M_2 = Z_4$*as Z-module.*
M_1*and* M_2 *are Ee-simple, take* $N = Z_{12} \bigoplus < \bar{2} > < M_1 \bigoplus M_2$ *, so there exists* $W =< \bar{2} > \bigoplus < \bar{2} >< Z_{12} \bigoplus < \bar{2} >$ *and* $W <_e M_1 \bigoplus M_2$*. That is N is not Ee-simple submodule . Hence* $M_1 \bigoplus M_2$*is not Ee-simple .*

5. *Fuzzy e-simple submodules, fuzzy Ee- simple modules and fuzzy e-maximal submodule*

In this section we fuzzify and study the concepts which are given in Section 4

Definition 5.1. *Let X be a fuzzy module over an R-module M.*

(i) Let $A < X$, $A \neq (0)_1$. A is called e-simple fuzzy submodule if there is no $B <_e X$ such that $B < A$

(ii) X is called Ee-simple fuzzy module if every nonzero proper fuzzy submodule of X is e-simple fuzzy submodule .

(iii) Let $A < X$ .A is called e-maximal fuzzy submodule if X/A is Ee-simple fuzzy module.

Proposition 5.2. *Let $(0)_1 \neq A < X$, A is e-simple if and only if A_* is e-simple in X_*.*
Proof . $\Longrightarrow$ *Since $(0)_1 \neq A < X$, then $(0\ ) \neq A_* < X_*$. Assume there exists $N <_e X_*$ and $N < A_*$. Definition $B : M \to [0,1]$ by:*

$$B(x) = \begin{cases} 1 & \text{if } x \in N \ , \\ 0 & \text{otherwise} \end{cases} .$$

Hence $B < X$ and $B_* = N$. Hence $B_* <_e X_*$ and $B_* < A_*$. Hence $B <_e X$ [4].
Thus A is not e-simple which is a contradiction.
$\Longleftarrow$ *To prove A is e-simple fuzzy submodule. Assume there exists $B <_e X$ and $B < X$. It follow that $B_* <_e X_*$ and $B_* < A_*$. By [4], Remark 2.1.4]. Thus A_* is not e-simple submodule of X_* which is a contradiction.* □

Corollary 5.3. *Let X be a fuzzy module over an R-module M, If X_* is Ee-simple module, then X is Ee-simple fuzzy module. The converse hold if condition (*) hold.*

Proposition 5.4. *Let X be a fuzzy module over an R-module M, let $A < X$, A is e-maximal fuzzy submodule if and only if $A_*$is e-maximal submodule in X_* Provided $X(x) = 1$, $\forall x \in M$.*
Proof . $\Longrightarrow$ *If A is e-maximal fuzzy submodule, then X/A is Ee-simple fuzzy module. By previous Proposition $(X/A)_*$is Ee-simple module. But by ([10] Remark 1.3.7) $(X/A)_* = X_*/A_*$ Hence X_*/A_* is Ee-simple module and so A_* is e-maximal submodule of X_*.*
$\Longleftarrow$ *If A_* is e-maximial submodule of X_*. Thus X_*/A_* is Ee-simple module.*
But $X\,(x) = 1$, so $X_*/A_* = (X/A)_*$ by [10].
Hence X_*/A_* is Ee-simple module and by pervious proposition X/A is Ee-simple fuzzy module. Therefore A is e-maximial fuzzy submodule □

5.1. Remarks and Examples

(i) It is clear that every semisimple fuzzy module is Ee-simple fuzzy module, but not conversely for example:
Let $X : Z_4 \Longrightarrow [0,1]$ defined by $X\,(x) = 1, \forall x \in Z_4$
Let $(0)_1 \neq A < X$, then $(0\ ) \neq A_* < X_* = Z_4$and so $A_* =< \bar{2} >$. Hence A_* is e- simple submodule in $X_* = Z_4$ Therefore A is e- simple submodule in X and X is Ee-simple fuzzy module. However X is not semisimple fuzzy module since

$$A(x) = \begin{cases} 1 & \text{if } x \in < \bar{2} > \ , \\ 0 & \text{otherwise} \end{cases} .$$

A is not direct summand of X .

(ii) If $A \leq X$, A is semimaximal, then A is e-maximal fuzzy submodule but the converse is not true.

Proof . *If A semimaximal fuzzy submodule, then $\frac{X}{A}$ is semisimple fuzzy module, hence by part (i) X/A is Ee-simple fuzzy module, and so A is is e-maximal fuzzy submodule.* □

Example: Let $X : Z \to [0,1]$ defined by $X(x) = 1, \forall x \in Z$. Let

$$A(x) = \begin{cases} 1 & if\, x \in 4Z \ , \\ 0 & otherwise \end{cases}.$$

$A_* = 4Z \Longrightarrow X_*/A_* \approx Z_4$, which is Ee-simple module. So A_* is e-maximal and hence A is e-maximal, by Proposition 4.4. But $X_*/A_* = (X/A)_*$ by [10]. Thus $(X/A)_* \approx 4Z$ is not semisimple. Therefore X/A is not semisimple by ([10], Lemma 2.1.7). Thus A is not semimaximal.

(iii) By combining above properties with basic properties in S.3, we have maximal fuzzy submodule $\Longrightarrow$ almost maximal fuzzy submodul $\Longrightarrow$ semimaximal fuzzy submodu $\Longrightarrow$ e-maximal fuzzy submodule, and e-maximal fuzzy submodule $\nRightarrow$ almost maximal fuzzy submodule $\nRightarrow$ semimaximal fuzzy submodule $\nRightarrow$ Maximal fuzzy submodule.

Proposition 5.5. *If A is e-simple of X, $A < B < X$, then B/A is e-simple fuzzy submodule in X/A.*

Proof . *Assume there exists $C/A < B/A$ and $C/A <_e X/A$. Hence $A < C < B$ and $C <_e< X$ [4] i.e $(A < C$ and $C <_e X)$.*
Thus A is not e-simple fuzzy submodule which is a contraction. □

Corollary 5.6. *If X is Ee-simple fuzzy module , $A < X$, then X/A is Ee-simple fuzzy module.*

Proof . *It is follows by Proposition 5.5.* □

Hint: *The converse of Corollary 5.6 may be not true, see the following example:*

Example 5.7. *Example: Let$X : Z_8 \to [0,1]$, let $X(x) = 1$.*
X is not Ee-simple fuzzy module, since there exists fuzzy submodule.

$$A(X) = \begin{cases} 1 & if\, x \in <\bar{2}> \ , \\ 0 & otherwise \end{cases}.$$

A is not e-simple fuzzy submodule. Since $X_*/A_* = (X/A)_*$ by [10].
Therefore $X_*/A_* = Z_8/<\bar{2}>\approx Z_2$ is Ee-simple module, hence $(X/A)_*$ is Ee- simple module and by Coroally 5.3, X/A is a fuzzy Ee-simple module .

Corollary 5.8. *If X is Ee-simple fuzzy module, then $\forall\ A < X$, A is e-maximal fuzzy submodule .*

Proof . *By Proposition 5.5, X is Ee-simple fuzzy module, then X/A is Ee-simple fuzzy module $\forall\ A < X$, then A is e-maximal fuzzy submodule.* □

Hint: *If $A < X$, $A \neq (0\ )_1$ such that A is e-simple , then it is necessary that A is e-maximal.*

Example 5.9. *Example: Let* $X: \ Z_8 \ \rightarrow [0,1]$*, such that* $X(x) = 1$*, and*

$$A(x) = \begin{cases} 1 & \text{if } x \in <\bar{4}> , \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$

A is e-simple fuzzy module, but A is not e-maximal, since there exists $B < \ X$*, such that*

$$B(x) = \begin{cases} 1 & \text{if } x \in <\bar{2}> , \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$

Hence $A < B$*. Moreover* $B <_e X$*.*

Corollary 5.10. *If* $f : X \rightarrow X'$ *be epimorphism with* $\ker f$ *is e-maximal fuzzy submodule, then* X' *is Ee- simple fuzzy module.*

Proof . *Since* $\ker f$ *is e-maximal fuzzy submodule, then* $X/\ker f$ *is Ee-simple fuzzy module. But* $X' \ \approx X/\ker f$*. Thus* X' *is Ee-simple fuzzy module.* □

Proposition 5.11. *If A is e-maximal of X . A < B, B is fuzzy submodule of X. Then B/A is e-maximial in X/A and B is e-maximal in X.*

Proof .*Consider* $(X/A)/(B/A) \approx X/B$*. But A is e-maximal e of X , then X/A is Ee-simple fuzzy module. Hence* $(X/A)/(B/A) \approx X/B$ *is Ee-simple by (Proposition 5.7) . Thus B/A is e-maximal fuzzy submodule in X/A. Also X/B is Ee-simple implies B is e-maximal in X.* □

Corollary 5.12. *If* $A, \ B \leq X$ *such that A is e-maximial, then* $A + B$ *is e-maximal.*

Proof . *It is Clear by Proposition 5.11.* □

Remark 5.13. *If* X_1 *and* X_2 *be Ee-simple fuzzy modules overr* M_1 *and* M_2 *respectively, then* $X_1 \bigoplus X_2$ *need not be Ee-simple fuzzy module.*

Example 5.14. *Let* $X_1(x) = 1, \ \ \forall x \in M_1 = Z_{12}$ *,* $X_2(x) = 1$ *,* $\forall x \in M_2 = Z_{12}, A : Z_{12} \bigoplus Z_{12} \rightarrow [0,1]$ *defined by:*

$$A(x,y) = \begin{cases} 1 & \text{if } x \in Z_{12}, \ \ y \in <\bar{2}> \\ 0 & \text{otherwise} \end{cases}.$$

$A_* = Z_{12} \bigoplus <\bar{2}>$ *which is note-simple fuzzy submodule in* $(X_1)_* \bigoplus (X_1)_* = (X_1 + X_2)_* = Z_{12} \bigoplus Z_{12}$*. Thus A is not e-simple fuzzy submodule in* $(X_1) \bigoplus (X_1)$ *(by Proposition 5.2).*

Remark 5.15. *Let Aand B be e-maximal fuzzy submodules of fuzzy module X .Then* $A \cap \ B$ *need not be e-maximal fuzzy submodule, as the following example shows:*

Let $X : Z \rightarrow [0,1] \quad X(x) = 1$*, Let* $A : Z \rightarrow [0,1]$ *,* $B : Z \rightarrow [0,1]$ *defined by:*

$$A(x) = \begin{cases} 1 & \text{if } x \in 4Z, \\ 0 & \text{otherwise} \end{cases}, \qquad B(x) = \begin{cases} 1 & \text{if } x \in 9Z, \\ 0 & \text{otherwise} \end{cases}, \qquad (A \cap B)(x) = \begin{cases} 1 & \text{if } x \in 36Z, \\ 0 & \text{otherwise} \end{cases}.$$

$(X/A \cap B)_* = X_*/(B \cap A)_* = Z/36Z \approx Z_{36}$.

But Z_{36} *is not Ee-simple module, since there exists* $N =<\bar{2}>< Z_{36}$ *and there exists* $W =<\bar{6}><<\bar{2}>$*. and* $<\bar{6}> <_e \ Z_{36}$ *. Thus* $(X/A \cap B)_*$ *is not e-simple, so* $(X/A \cap B)$ *is not e-simple by (Proposition 5.4). Thus* $A \cap B$ *is not e-maximal.*

Note that $(X/A)_* = X_*/A_* \approx Z_4$ is Ee-simple module. So A is e- maximal.
$(X/A)_* = X_*/A_* \approx Z_9$ is Ee-simple. So B is e- maximal.

Proposition 5.16. *Let be X a fuzzy module of an R- module M. The following assertion are equivalent :*

(i) $(0)_1$ *is e-maximal.*
(ii) Every proper fuzzy submodule of X is e- maximal.
(iii) X is Ee-simple.

Proof .
(i) $\Longrightarrow$ *(iii) Since* $(0)_1$ *is e- maximal, then* $X/(0)_1$ *is Ee- simple fuzzy module. Thus X is Ee-simple.*
(iii) $\Longrightarrow$ *(i) X is Ee-simple, that is* $X/(0)_1 \approx X$ *Ee- simple. Thus* $(0)_1$ *is e- maximal.*
(ii) $\Longrightarrow$ *(iii) It follows by Corollary 5.10.*
(iii) $\Longrightarrow$ *(ii) By (ii),* $(0)_1$ *is e-maximal, so* $X/(0)_1 \approx X$ *is Ee-simple fuzzy module.* □

Open Problems

(i) Let X be a fuzzy module over R-module M .
Define F-Rad $X_\alpha = \cap$ all fuzzy almost maximal submodules. Suggest kind of fuzzy small submodules.

(ii) Let X be a fuzzy module over R-module M
F- Soc(X)= $\cap$ all fuzzy essential submodules of X, studying this concept.
Is F-Soc(x)=$\sum$ all fuzzy simple submodules of X

(iii) It is known that Z_2 (M), the Z_2 –torisionsubmodules of an R-module M, where $Z_2(M)/Z(M) = Z(M/Z(M))$, where $Z(M)$ is the singular submodules of M.
Fuzzifing and studying the concepts $Z(M)$ and $Z_2(M)$

6. Conclusion

Most of the results about almost maximal submodules transferred to fuzzy almost maximal submodules . However some results needed some extra conditions.

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