ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 40 (2014), No. 6, pp. 1441–1451

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 40 (2014), No. 6, pp. 1441–1451
Online ISSN: 1735-8515

ON SPECIAL SUBMODULE OF MODULES

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(Communicated by Teo Mora)

Abstract. Let R be a domain with quotiont field K, and let N be a submodule of an R-module M. We say that N is powerful (strongly primary) if $x, y \in K$ and $xyM \subseteq N$, then $x \in R$ or $y \in R$ ($xM \subseteq N$ or $y^n M \subseteq N$ for some $n \geq 1$). We show that a submodule with either of these properties is comparable to every prime submodule of M, also we show that an R-module M admits a powerful submodule if and only if it admits a strongly primary submodule. Finally we study finitely generated torsion free modules over domain each of whose prime submodules are strongly primary.
Keywords: Prime submodule, strongly prime submodule, primary submodule, strongly prime submodule, power submodule.
MSC(2010): Primary: 65F05; Secondary: 46L05, 11Y50.

1. Introduction

Throughout this work R will denote an integral domain with quotient field K. Recall from [3] that a prime ideal P of R is said to be strongly prime if, $xy \in P$ for elements $x, y \in K$, we have $x \in P$ or $y \in P$. In this paper, we consider two generalizations of this concept . We define a non-zero submodule N of an R-module M to be powerful if, $xyM \subseteq N$ for elements $x, y \in K$, we have $x \in R$ or $y \in R$. It is easy to see that R is powerful if and only if R is a valuation domain. In the first section, we show that a powerful prime submodule is strongly prime. We also show that if N is a proper powerful submodule of M, then $\sqrt{(N : M)}$ is a prime ideal in general and strongly prime when R is seminormal. Moreover, a powerful submodule N is comparable to every non-zero

Article electronically published on December 11, 2014.
Received: 26 December 2012, Accepted: 19 October 2013.
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prime submodule of M, from which it follows that the prime submodule contained in $\sqrt{(N:M)}$ is linearly ordered.

As another generalization of the notion of "strongly prime" in Sec.2, we define a submodule N of M to be strongly prime if, $xyM \subseteq N$ with $x, y \in K$, we have $xM \subseteq N$ or $y^n M \subseteq N$ for some $n \geq 1$. Simple examples show that "powerful" and strongly primary are different notions. A strongly proper primary submodule of M is clearly primary, and we observe that the converse is true over a valuation domain.

2. **Powerful submodules**

In this section, we want to consider the concept of a powerful submodule which is a generalization of the concept of strongly prime submodule. We shall prove that when a submodule N is prime, N is strongly prime if and only if N is a powerful submodule.

Definition 2.1. *Let R be an integral domain with quotient field K and M be an R-module. We define a non-zero submodule N of M to be powerful if, $xyM \subseteq N$, for elements $x, y \in K$, we have $x \in R$ or $y \in R$.*

We begin with a simple, but useful theorem on the concept of a powerful submodule.

Theorem 2.2. *A submodule N of M is powerful, if and only if $x^{-1}(N:M) \subseteq R$ for every $x \in K \setminus R$.*

Proof. Assume that N is powerful, and take $x \in K \setminus R$. Then, for $a \in (N:M)$ we have, $xx^{-1}a = a \in (N:M)$, whence $x^{-1}a \in R$. For the converse, take $yzM \subseteq N$, $y, z \in K$. Suppose $y \notin R$. Then, $z = y^{-1}yz \in y^{-1}(N:M) \subseteq R$, as desired. □

Corollary 2.3. *R is powerful if and only if R is a valuation domain.*

Theorem 2.4. *Let N be a powerful submodule of an R-module M and Q be a proper submodule of N. Then $\frac{N}{Q}$ is a powerful submodule of $\frac{M}{Q}$. In particular, if $P = (Q:M)$ is a prime ideal of R, then $\frac{N}{Q}$ is a powerful $\frac{R}{P}$-submodule of $\frac{M}{Q}$.*

Proof. Since we have $(\frac{N}{Q} : \frac{M}{Q}) = Ann(\frac{\frac{M}{Q}}{\frac{N}{Q}}) \cong Ann(\frac{M}{N}) = (N:M)$, and N is a powerful submodule, it follows that $\frac{N}{Q}$ is a powerful submodule of $\frac{M}{Q}$ as R-module. □

Definition 2.5. *A submodule N of an R-module M is said to be prime in case $am \in N$, where $m \in M$ and $a \in R$, implies that $m \in N$ or $a \in (N : M)$. Also, N is said to be a strongly prime submodule in case N is a prime submodule and whenever, $xy \in (N : M)$ for elements $x, y \in K$, we have $x \in (N : M)$ or $y \in (N : M)$.*

Theorem 2.6. *Let N be a prime submodule of an R-module M. Then, N is a strongly prime submodule if and only if N is a powerful submodule of M.*

Proof. Suppose that N is a strongly prime submodule. If $x \in K \setminus R$ and $n \in (N : M)$ then $n = nxx^{-1} \in (N : M)$, whence $nx^{-1} \in (N : M)$ or $x \in (N : M)$. Since $x \notin R$ we must have $nx^{-1} \in (N : M)$. Thus $x^{-1}(N : M) \subseteq (N : M)$.

For the converse, let N be a prime and powerful submodule of M and $xy \in (N : M)$ whenever $x, y \in K$. Since $(N : M)$ is an ideal of R, hence $x^2y^2 \in (N : M)$. We may assume that $x \notin R$ and $y \in R$. If $x^2 \in R$, then since $x \notin R$, $x^2 \notin (N : M)$, and by the fact that $x^2y^2 \in (N : M)$ it follows that $y^2 \in (N : M)$, whence $y \in (N : M)$. If $x^2 \notin R$, Then, since $(\frac{y^2}{xy})x^2 \in (N : M)$, we have $\frac{y^2}{xy} \in R$. Hence $y^2 = (\frac{y^2}{xy})xy \in (N : M)$, and again we have $y \in (N : M)$. □

Lemma 2.7. *Let $Q \prec N$ be a submodule of an R-module M and N be a strongly prime submodule of M. Then $\frac{N}{Q}$ is a powerful submodule of $\frac{M}{Q}$.*

Proof. Use Theorem 2.6 and 2.4. □

Lemma 2.8. *If $Q \prec N$ is a submodule of an R-module M and N is a strongly prime, then Q is powerful submodule of M.*

Proof. We have $(Q : M) \subseteq (N : M) \subseteq R$, therefore $xy \in (Q : M)$ for elements $x, y \in K$, implies that $xy \in (N : M)$. Since N is strongly prime, $x \in (N : M)$ or $y \in (N : M)$, whence $x \in R$ or $y \in R$, and it follows that Q is a powerful submodule of M. □

Corollary 2.9. *If $Q \prec N$ is a submodule of an R-module M and N is powerful, then Q is also powerful.*

Proof. Since $(Q : M) \subseteq (N : M)$, $xy \in (Q : M)$ implies that $xy \in (N : M)$. Now since N is powerful, we have $x \in R$ or $y \in R$. □

Theorem 2.10. *If $Q \prec N$ is a prime submodule of an R-module M and N is strongly prime then, Q is strongly prime.*

Proof. This follows from Theorem 2.6 and Corollary 2.9. □

Theorem 2.11. *Let N be a powerful submodule of an R-module M and Q an arbitrary submodule of M, then we have:*

(1) $(Q : M) \subseteq (N : M)$ *or* $(N : M)^2 \subseteq (Q : M)$.
(2) *If Q is a prime submodule of an R-module M, then $(N : M)$ and $(Q : M)$ are comparable.*

Proof. To prove (1), we suppose that Q is a submodule of M and $(Q : M) \nsubseteq (N : M)$. Choose $a \in (Q : M) \setminus (N : M)$ and take $b, c \in (N : M)$. Then, $(\frac{bc}{a})(\frac{a}{b}) \in (N : M)$, and since N is powerful with $\frac{a}{b} \notin R$, we have $\frac{bc}{a} \in R$. Hence $bc \in aR \subseteq (Q : M)$, as desired.

Statement (2): Since Q is a prime submodule, hence $(Q : M)$ is a prime ideal in R. By (1), we have $(Q : M) \subseteq (N : M)$ or $(N : M)^2 \subseteq (Q : M)$. Since $(Q : M)$ is a prime ideal, $(Q : M) \subseteq (N : M)$ or $(N : M) \subseteq (Q : M)$. □

Definition 2.12. *Let R be an integral domain and M a torsion free R-module. Then, M is said to be Fully strongly prime (FSP) if each prime submodule of M is strongly prime.*

Theorem 2.13. *Let R be an integral domain and M be a finitely generated torsion free R-module, then M is a FSP -module if and only if some maximal submodule of M is powerful.*

Proof. Let M be a FSP-module and N be a maximal submodule of M. Therefore N is a prime submodule and by Theorem 2.6 N is powerful.

For the converse let some maximal submodule of M as N be a powerful submodule and also suppose that Q is an arbitrary non-zero prime submodule of M. Now by 2.11(2), we have $(Q : M) \subseteq (N : M)$ or $(N : M) \subseteq (Q : M)$. Now if $(Q : M) \subseteq (N : M)$, then by Corollary 2.9, Q is powerful and therefore by 2.6, Q is strongly prime. And if $(N : M) \subseteq (Q : M)$, Since $(N : M)$ a maximal ideal in R, it follows that $(N : M) = (Q : M)$. Thus if $xy \in (Q : M)$ for every $x, y \in K$, then $xy \in (N : M)$. So $x \in (N : M)$ or $y \in (N : M)$ and hence $x \in (Q : M)$ or $y \in (Q : M)$. Therefore Q is a strongly prime submodule of M. □

Lemma 2.14. *Let $\{N_\alpha\}$ be a family of powerful ideals of R, then $\sum N_\alpha$ is a powerful ideal.*

Proof. Use Theorem 2.2 □

Question: If $\{N_\alpha\}$ is a family of powerful submodules of an R-module M, is then $\sum N_\alpha$ a powerful submodule of M?

Theorem 2.15. *If R contains a powerful ideal, then R contains the largest unique powerful ideal.*

Proof. Let $\{N_\alpha\}$ be all powerful ideals of R, then $\sum N_\alpha$ is the largest unique powerful ideal of R by Lemma 2.14. □

Theorem 2.16. *If N is a proper powerful submodule of R-module M, then $P = \cap_k (N : M)^k$ is a strongly prime ideal.*

Proof. By Theorem 2.6 and Corollary 2.9, it suffices to show that P is a prime ideal. Take $xy \in P$ with $x \notin P$. Then $x \notin (N : M)^n$ for some $n > 0$, whence by theorem 2.11 (1), $(N : M)^2 n \subseteq xR$. Hence for each $k > 0$, we have $xy \in P \subseteq (N : M)^{2n+k} \subseteq x(N : M)^k$. Thus $y \in (N : M)^k$ for each $k > 0$. It follows that $y \in P$. □

Theorem 2.17. *Let N be a powerful submodule of an R-module M. If $x, y \in K$ and $xy \in \sqrt{(N : M)}$, then there is a positive integer m such that either $x^m \in (N : M)$ or $y^m \in (N : M)$. In particular, if N is a proper powerful submodule of an R-module M, then $\sqrt{(N : M)}$ is a prime ideal in R.*

Proof. Let $xy \in \sqrt{(N : M)}$, then $(xy)^n \in (N : M)$ for some $n > 0$. Hence $(\frac{x^{3n}}{x^n y^n})(\frac{y^{3n}}{x^n y^n}) = x^n y^n \in (N : M)$. Since N is powerful, so that either $\frac{x^{3n}}{x^n y^n} \in R$ or $\frac{y^{3n}}{x^n y^n} \in R$, either $x^{3n} \in (N : M)$ or $y^{3n} \in (N : M)$. □

In spite of Theorem 2, the radical of a powerful ideal need not be powerful, as the following example shows.

Example 2.18. *Let $V = K + M$ be a discrete valuation domain with dim V=1, where K is a field and $M = tV$ is a maximal ideal of V, and let $R = K + M^2$.*
Claim: M^3 is a powerful ideal of R. To see this, take $xy \in M^3$, with $x, y \in K$ (the common quotient field of R and V). We may write $x = ut^n$, $y = vt^m$, where u, v are units of V and n, m are integers. Since $xy \in M^3$, we must have $n + m \geq 3$. Hence either $n \geq 2$ or $m \geq 2$, say $n \geq 2$. Then $x = ut^n \in M^2 \subseteq R$. This proves the claim. However Rad$(M^3) = M^2$ is not powerful since $t^2 \in M^2$ but $t \notin R$.

Corollary 2.19. *If N is a powerful submodule of an R-module M, then $\sqrt{(N : M)}$ is a powerful ideal.*

Remark 2.20. *We prove below that, over a seminormal domain, the radical of a powerful submodule is powerful. First, we need a lemma.*

Lemma 2.21. *Let N be a powerful submodule of an R-module M. If $x \in K$ and $x^n \in (N : M)$ for some $n > 0$, then $x^{n+k} \in R$ for each $k \geq 0$.*

Proof. take $e = min\{m \geq 1 | x^m \in R\}$. Let k be a positive integer and write $k = qe + r$ with $0 \leq r < e$. If $r = 0$, then it is easy to see that $x^{n+k} \in R$. Suppose that $r > 0$. We have $x^{e-r}x^{qe+n+r} = x^n x^{(q+1)e} \in (N : M)$. Since $x^{e-r} \notin R$, we have $x^{n+k} = x^{qe+n+r} \in R$ as desired. □

Definition 2.22. *Let M be an R-module and N be a submodule of M. We say that N is radical if $x \in R$, $m \in M$ and $x^n m \in N$ for some $n > 0$ imply that $xm \in N$. A radical submodule N of M is said to be strongly radical, if $x \in K$, $m \in M$ and $x^n m \in N$ for some $n > 0$, imply that $xm \in N$. Also, R is called seminormal if $x \in R$ whenever, $x^n \in R$ for all sufficiently large n.*

Theorem 2.23. *Let N be a proper powerful submodule of an R-module M. Then $\sqrt{(N : M)}$ is a powerful ideal (and therefore strongly prime) if and only if $\sqrt{(N : M)}$ is strongly radical. In particular, if R is seminormal, then $\sqrt{(N : M)}$ is strongly prime.*

Proof. It is easy to see that a powerful radical submodule must be strongly radical. Suppose that $\sqrt{(N : M)}$ is strongly radical, and take $xy \in \sqrt{(N : M)}$ with $x, y \in K$. Then by Theorem we have $x^m \in (N : M)$ or $y^m \in (N : M)$ for some $m > 0$. We may suppose that $x^m \in (N : M)$. Then, $x^m \in \sqrt{(N : M)}$, whence $x \in R$, as desired. The "in particular" □

Statement now follows from Lemma .

Theorem 2.24. *Let R be an integral domain and $K = S^{-1}R$ ($S = R - 0$), then R is a valuation domain if and only if R as an ideal of R is powerful.*

Proof. Take R as a valuation domain and $xy \in R$ for $x, y \in K$, if $x \notin R$ then $x^{-1} \in R$. Therefore $x^{-1}xy \in R$, and hence $y \in R$. For the converse, we have $1 = x^{-1}x \in R$, hence $x \in R$ or $x^{-1} \in R$. □

Theorem 2.25. *Let T be a domain, R a subring of T, and I a powerful ideal of R. Then IT is a powerful ideal of T. In particular if $IT = T$ then T is a valuation domain.*

Proof. Suppose that $x \in K - T$, then $x \notin R$ and therefore $x^{-1}I \subseteq R$ and hence $x^{-1}IT \subseteq T$, so IT is a powerful ideal of T. Now if $IT = T$, then $x^{-1}T \subseteq T$ implies that T is valuation domain. □

Remark 2.26. *Let I be a powerful ideal of R and suppose that $P \subseteq I$ is a non-zero finitely generated prime ideal of R. Then R is a FSP-module as an R-module.*

Proof. If P is not maximal, then R contains a non-unit x with $x \notin P$. Since P is strongly prime (by Theorem 2.6 and Corollary 2.9) and $xx^{-1}P \subseteq P$ with $x \notin P$, we have $x^{-1}P \subseteq P$. Since $x^{-1}P \subseteq R$, Hence x^{-1} is integral over R, which is impossible. Thus P is maximal, and it follows that R is a FSP- module by Theorem 2.13. □

Corollary 2.27. *Let I be a powerful ideal and $m = \sqrt{I}$ be a maximal ideal of R. Then R is a local ring with maximal ideal m.*

Proof. It follows from Theorem 2.11. □

Corollary 2.28. *Let R be an integral domain, I a powerful ideal of R and $R \subseteq R'$ where R' is an overing of R such that $R' \neq K$ ($K = S^{-1}R$, $S = R - \{0\}$). Then*

(1) *If $IR' = R'$, then $P = N \cap R$, where N is the maximal ideal of R', is a common ideal which is powerful in both rings.*
(2) *If $IR' \neq R'$, then I^2R' is a common ideal, and I^3R' is powerful in both rings.*

Proof. For (1), recall that R' is a valuation domain by Theorem 2.25. By Theorem 2.11 I is comparable to P. The fact $IR' = R'$, then implies that $P \subseteq I$, whence P is powerful, and therefore strongly prime in R. Note that PR' is powerful in R' by Theorem 2.25. We claim that $PR' = P$, to verify this, let $x \in R' - R$. Clearly $x^{-1} \notin P$. Hence, since $x^{-1}xP \subseteq P$ and P is strongly prime, we have $xP \subseteq P$, as claimed.

(2) Let $x \in R' - R$. Then, by hypothesis, $x^{-1} \notin I$, whence $I^2 \subseteq x^{-1}R$ by Theorem 2.11. Hence, again $xI^2 \subseteq R$. Thus I^2R' is an ideal of R. Since $I^3R' \subseteq I$, I^3R' is powerful in R by Theorem 2.9, and I^3R' is powerful in R' by Theorem 2.25. □

Corollary 2.29. *Suppose that R' is an overing of R and that R and R' share the non-zero ideal J. If J is powerful in R', then J^3 is a powerful ideal of R.*

Proof. Let $x \in K - R$. If $x \notin R'$, then $x^{-1}J \subseteq R'$ by Theorem 2.2. In this case , we have $x^{-1}J^3 \subseteq J^2R' \subseteq R$. Now assume $x \in R'$. Since $x \notin J$, we have $J^2 \subseteq xR'$ by Theorem 2.11. Hence $x^{-1}J^3 \subseteq JR' = J \subseteq R$, and the proof is complete. □

Remark 2.30. *In Corollary 2.29, if R' is a valuation domain, then J^2 is powerful in R. However, for general, R', the third power is best possible, as the following example shows.*

Example 2.31. *Let $K = Q(\sqrt{2})$ and $V = K[[X]] = K + M$, $M = XK[[X]]$. Then let $R' = Q + M$, $J = XR'$, and $R = Q + J$. Then R and R' share the ideal J, and since R' is a FSP, J is powerful in R'. However, J^2 is not powerful in R, since $\sqrt{2}X \times \sqrt{2}X = 2X^2 \in J^2$, but $\sqrt{2}X \notin R$.*

3. **Strongly primary submodules**

In this section we shall show that if R is a seminormal ring, then strongly primary submodules are powerful and if N is a strongly primary submodule, then $(N : M)$ is comparable to every radical ideal of R.

Definition 3.1. *A submodule N of an R-module M is said to be primary in case $am \in N$, where $m \in M$, $a \in R$, implies that $m \in N$ or $a^n \in (N : M)$ for some positive integer number $n \geq 1$. Also N is said to be a strongly primary submodule in case N is a primary submodule and if $xy \in (N : M)$ for elements $x, y \in K$, we have $x \in (N : M)$ or $y^n \in (N : M)$ for some positive integer number $n \geq 1$.*

Theorem 3.2. *Let M be an R-module and R be a valuation domain. Then a primary submodule of M is strongly primary.*

Proof. Let K be a quotient field of R and N be a primary submodule of M. Also let $x, y \in K$ with $xy \in (N : M)$, and suppose that $x \notin (N : M)$. If $x \notin R$, then $x^{-1} \in R$, and we have $y = x^{-1}xy \in (N : M)$. Hence we may as well assume that $x \in R$. Since $x = y^{-1}xy \notin (N : M)$, it follows that $y \in R$. Now , since $x, y \in R$ with N primary, we have $y^n \in (N : M)$ for some positive integer $n \geq 1$, as desired. □

Remark 3.3. *Professor R. Gilmer ([3], Exercise 2, P.293), showed that if R is a valuation domain with $dimR > 1$, then there are ideals which are not primary. Since every ideal of a valuation domain is powerful, this shows that powerful ideals need not be (strongly) primary. Conversely, strongly primary ideals need not be powerful: In Example 2.18, M^2 is strongly primary but not powerful in R.*

Notation 3.4. *For a subset S of an R-module M, we define $E(S)$ by $E(S) = \{x \in K | (\forall n \geq 1)(\exists m \in M) x^n m \notin S\}$.*

Lemma 3.5. *A non-zero primary submodule N of M is strongly primary if and only if $x^{-1}N \subseteq N$ for each $x \in E(N)$.*

Proof. If N is strongly primary and $x \in E(N)$, then the equation $xx^{-1}N = N$ implies that $x^{-1}N \subseteq N$. Conversely, if $yz \in (N : M)$ with $y, z \in K$ and $z \in E(N)$, then the hypothesis yields $y = z^{-1}yz \in z^{-1}(N : M) \subseteq (N : M)$, as desired. □

Theorem 3.6. *Let R be a seminormal domain. If N is a proper strongly primary submodule of an R-module M, then N is powerful, and $\sqrt{(N : M)}$ is strongly prime. In particular, a prime submodule of M is strongly prime if and only if it is strongly primary.*

Proof. Let $x \in K - R$. We shall show that $x^{-1}(N : M) \subseteq (N : M)$ (whence $x^{-1}(N : M) \subseteq R$). By Lemma 3.5, it suffices to show that $x^n \notin (N : M)$ for all $n \geq 1$. Suppose, on the contrary, that $x^r \in (N : M)$, with r minimal. It is then easy to see that $x^{-k} \notin (N : M)$ for each $k \geq 0$, that is $x^{-1} \in E(N : M)$. By Lemma 3.5, this implies that $x^{r+1} = xx^r \in x(N : M) \subseteq (N : M)$. By induction, we get $x^t \in (N : M) \subseteq R$ for each $t \geq r$. However, the seminormality of R then implies that $x \in R$, a contradiction. □

Theorem 3.7. *Let N be a proper strongly primary submodule of an R-module M, and let R' be an overing of R. Then either $(N : M)R' = R'$ or $(N : M)R' = (N : M)$.*

Proof. Assume that $(N : M)R' \neq R'$ and pick $x \in R' - R$. If $x^{-n} \in (N : M)$ for some $n \geq 1$, then since $(N : M)R' \neq R'$, x^{-n} is a non-unit of R', a contradiction. Hence $x^{-1} \in E(N : M)$, and we have $x(N : M) \subseteq (N : M)$ by Lemma 3.5. Thus $(N : M)R' = (N : M)$. □

Corollary 3.8. *Let R' be an integral closure of the domain R and N be a proper strongly primary submodule of an R- module M, then $(N : M)R' = (N : M)$. Moreover, $(N : M)^3$ is powerful in both R and R'.*

Proof. The first conclusion follows from Theorem 3.7 and the lying over property of integral extensions. Since $(N : M)$ is automatically strongly primary in M, N is powerful in M by Theorem 3.6. It follows that $(N : M)^3$ is powerful in R and R' by Corollary 2.29. □

Corollary 3.9. *If N is a proper strongly primary submodule of an R-module M, then $\bigcap(N : M)^n$ is a strongly prime ideal of R.*

Proof. This follows from Theorem 2.16 and the fact that $(N : M)^3$ is powerful. □

Corollary 3.10. *If N is a strongly primary submodule of an R-module M, then $(N : M)$ is comparable to every radical ideal of R. Moreover, the prime submodules of M which are properly contained in N are strongly prime and linearly ordered.*

Proof. Let J be a radical ideal of R, and suppose that $(N : M) \nsubseteq J$. Choose $a \in (N : M) - J$, and $b \in J$. Since $(\frac{a^2}{b})(\frac{b}{a}) = a \in (N : M)$ and $\frac{a^2}{b} \in E(R) \subseteq E(N : M)$, we have $\frac{b}{a} \in (N : M)$. Hence $J \in (N : M)$, as desired.
If Q is a prime submodule which is properly contained in N, then, since $(N : M)^3$ is powerful and $(Q : M) \subseteq (N : M)^3$, Q is also powerful. Then Q is strongly prime. □

Corollary 3.11. *If N is a prime submodule of M which is strongly primary but not strongly prime, then N is the only prime with this property.*

Corollary 3.12. *Let R be an integral domain, M be an R-module, also R-module $M' \neq S^{-1}M$ $(S = R - 0)$ be an overing of M, that is $M \subseteq M'$ and N be a strongly primary submodule of M. Then we have the following cases:*

1. *If $(N : M)M' \neq M'$, then $(N : M)M' = N$ is a common strongly primary submodule.*
2. *If $(N : M)M' = M'$, then M' is strongly primary, and for each maximal submodule N' of M', $N \cap M$ is a common strongly prime submodule of M and M'.*

Proposition 3.13. *Let N be a strongly primary submodule of M. Then:*

1. *$(N : M) \subseteq xR$ for every $x \in R \setminus \sqrt{(N : M)}$.*
2. *if $(N : M)$ is finitely generated, then R is quasilocal with maximal ideal $\sqrt{(N : M)}$.*

Proof. Let $x \in R \setminus \sqrt{(N : M)}$. Then $x \in E(N : M)$ and so (by Lemma 3.5) $x^{-1}(N : M) \subseteq (N : M)$. Hence $(N : M) \subseteq x(N : M) \subseteq xR$, proving (1).

(2) The relation $x^{-1}(N : M) \subseteq (N : M)$ shows that x^{-1} is integral over R. Since $x \in R$, we have $x^{-1} \in R$. It follows that R is quasilocal with maximal ideal $\sqrt{(N : M)}$. □

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