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Direct and fixed point methods approach to the generalized Hyers–Ulam stability for a functional equation having monomials as solutions

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Abstract

The main goal of this paper is the study of the generalized Hyers-Ulam stability of the following functional equation $f(2x+y)+f(2x-y)+(n-1)(n-2)(n-3)f(y)=2^{n-2}[f(x+y)+f(x-y)+6f(x)]$ where $n=1,2,3,4$, in non–Archimedean spaces, by using direct and fixed point methods.

Keywords: Hyers- Ulam stability; non -Archimedean normed space; p - adic field

1. Introduction

A classical question in the theory of functional equations is the following: *when is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?*

If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940.

In the next year, D. H. Hyers [2] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces.

In 1978, Th. M. Rassias proved a generalization of Hyers` theorem for additive mappings. The result of Th. M. Rassias has influenced the development of what is now called the Hyers-Ulam-Rassias stability theory for functional equations.

Theorem 1. ([3]): Let $f:E\to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon(\|x\|^p+\|y\|^p)$$

for all $x,y\in E$ where ε and p are constants with $\varepsilon>0$ and $0\le p<1$. Then the limit

$$L(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

exists for all $x\in E$ and $L:E\to E'$ is the unique additive mapping which satisfies

$$\|f(x)-L(x)\|\le\frac{2\varepsilon}{2-2p}\|x\|^p$$

for all $x\in E$. Also, if for each $x\in E$ the function $f(tx)$ is continuous in $t\in R$, then L is linear.

In 1994, a generalization of Rassias' theorem was obtained by Gavruta [4] by replacing the bound $\varepsilon(\|x\|^p+\|y\|^p)$ with a general control function $\phi(x,y)$.

Let X and Y be vector spaces and let $f:X\to Y$ be a mapping for each $n=1,2,3$, consider the functional equation

$$f(2x+y)+f(2x-y)=2^{n-2}[f(x+y)+f(x-y)+6f(x)] \quad (1)$$

Also, consider the functional equation

$$f(2x+y)+f(2x-y)+6f(y)=4[f(x+y)+f(x-y)+6f(x)] \quad (2)$$

For $X=Y=R$, the monomial $f(x)=cx^n$ is a solution of (1) for each $n=1,2,3$ and the monomial

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$f(x)=cx^4$ is a solution of (2). It is easy to show that, a mapping $f:X\to Y$ satisfies (1) for $n=1$ if and only if it also satisfies the Cauchy functional equation $f(x+y)=f(x)+f(y)$.

For $n=2$, in [5] it was shown that the equation (1) is equivalent to the quadratic functional equation.

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

In 2002, Jun and Kim [6] solved the functional equation (1) for $n=3$. In 2003, Chung and Sahoo [7] introduced the quartic equation

$$\begin{aligned} &f(x+2y)+f(x-2y)+6f(x)\\ &=4\left[f(x+y)+f(x-y)+6f(y)\right]. \end{aligned} \tag{3}$$

In [8], the equation (2) was shown to be equivalent to the above equation.

In 1897, Hensel [9] introduced a normed space which does not have the Archimedean property.

In this paper, the generalized Hyers-Ulam stability of functional equation

$$\begin{aligned} &f(2x+y)+f(2x-y)+(n-1)(n-2)(n-3)f(y)=\\ &2^{n-2}\left[f(x+y)+f(x-y)+6f(x)\right]. \end{aligned} \tag{4}$$

will be investigated in non- Archimedean normed space.

In [8], Bae and Park obtained the general solution of the functional equation (4) and proved the generalized Hyers-Ulam stability of this functional equation in Banach * -algebra.

Remark 1. For convenience, for all x, y, let

$$\begin{aligned} &\Omega_f^n(x,y)=f(2x+y)+f(2x-y)+\\ &(n-1)(n-2)(n-3)f(y)-\\ &2^{n-2}\left[f(x+y)+f(x-y)+6f(x)\right] \end{aligned}.$$

2. Preliminaries

Definition 1. By a non-Archimedean field, we mean a field K equipped with a function (valuation): $K\to[0,\infty)$ such that for all $r,s\in K$, the following conditions hold:

$(i)\ |r|=0$ if and only if $r=0$

$(ii)\ |rs|=|r||s|$

$(iii)\ |r+s|\le\max\{|r|,|s|\}.$

Definition 2. Let X be a vector space over a scalar field K with a non–Archimedean non-trivial valuation. A function $\|.\|:X\to R$ is a non–Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$

(ii) $\|rx\|=|r|\|x\|(r\in K, x\in X)$

(iii) the strong triangle inequality (ultra-metric), namely

$\|x+y\|\le\max\{\|x\|,\|y\|\}. \qquad x,y\in X$

Then $(X,\|.\|)$ is called a non- Archimedean space.

Due to the fact that

$$\|x_n-x_m\|\le\max\{\|x_{j+1}-x_j\|; m\le j<n\} \qquad (n>m)$$

Definition 3. A sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non–Archimedean space. By a complete non–Archimedean space, that is, one in which every Cauchy sequence is convergent.

The most important examples of non–Archimedean spaces are $p-$adic numbers. A key property of $p-$adic numbers is that they do not satisfy the Archimedean axiom: for all $x,y>0$, there exists an integer n such that $x<ny$.

Example 1. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x\in z$ such that $x=\frac{a}{b}p^{n_x}$ where a and b are integers not divisible by p. Then $|x|_p:=p^{-n_x}$ defines a non–Archimedean norm on Q. The completion of Q with respect to the metric $d(x,y)=|x-y|_p$ is denoted by Q_p which is called the $p-$adic number field. In fact, Q_p is the set of all formal series $x=\sum_{k\ge n_x}^{\infty}a_kp^k$ where $|a_k|\le p-1$ are integers. The addition and multiplication between any two elements of Q_p are defined naturally. The norm $\left|\sum_{k\ge n_x}^{\infty}a_kp^k\right|_p=p^{-n_x}$ is a non–Archimedean norm on Q_p and it makes Q_p a locally compact filed.

Definition4. Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a generalized metric on X if d satisfies the following conditions:

$(i)\ d(x,y)=0$ if and only if $x=y$, for all $x,y\in X$;

$(ii)\ d(x,y)=d(y,x)$ for all $x,y\in X$;

$(iii)\, d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Theorem 2. Let (X, d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$; either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

$(i)\ d(J^n x, J^{n+1} x) < \infty$ for all $n \ge n_0$;

(ii) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(iii) y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;

$(iv)\ d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

3. Non–Archimedean stability of functional equation (4): direct method

Throughout this section, we assume that G is an additive semi-group and X is a complete non–Archimedean space.

Remark 2. For convenience, for each $n = 1, 2, 3, 4,$ let

$$a_n = \frac{|(n-1)(n-2)(n-3)|}{|2 + (n-1)(n-2)(n-3) - 2^{n+1}|}$$

Theorem 3. For each $n = 1, 2, 3, 4,$ let $\varsigma_n : G^2 \to [0, +\infty)$ be a function such that

$$\lim_{m \to +\infty} \frac{\varsigma_n(2^m x, 2^m y)}{|2|^{mn}} = 0 \tag{5}$$

for all $x, y \in G$. Let for each $x \in G$ the limit

$$\Omega(x) = \lim_{m\to\infty} \max\left\{ \frac{\varsigma_n(2^k x, 0)}{|2|^{kn}}, \frac{a_n \varsigma_n(0,0)}{|2|^{kn}};\ 0 \le k \le m \right\} \tag{6}$$

exists. Suppose that $f : G \to X$ be mapping satisfying the inequality

$$\left\| \Omega_f^n(x, y) \right\| \le \zeta_n(x, y) \tag{7}$$

for all $x, y \in G$. Then the limit

$$\vartheta(x) := \lim_{m\to\infty} \frac{f(2^m x)}{2^{mn}}$$

exists for all $x \in G$ and $\vartheta(x) : G \to X$ is a mapping satisfying

$$\| f(x) - \vartheta(x) \| \le \frac{1}{|2|} \Omega(x) \tag{8}$$

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty} \lim_{m\to\infty} \max\left\{ \frac{\zeta_n(2^k x, 0)}{|2|^{kn}}, \frac{a_n \zeta_n(0,0)}{|2|^{kn}};\ j \le k < m + j \right\} = 0$$

Then $\vartheta(x)$ is the unique mapping satisfying (8).

Proof: Letting $x = y = 0$ in (7), we get

$$\| f(0) \| \le \frac{\zeta_n(0,0)}{|2 + (n-1)(n-2)(n-3) - 2^{n+1}|} \tag{9}$$

Putting $y = 0$ in (7), we get

$$\left\| 2f(2x) + (n-1)((n-2)(n-3) f(0) - 2^{n+1} f(x) \right\| \le \zeta_n(x, 0) \tag{10}$$

for all $x \in G$. By the above two inequalities, we have

$$\begin{aligned} \| 2f(2x) - 2^{n+1} f(x) \| &= \| 2f(2x) \pm (n-1)(n-2)(n-3) f(0) - 2^{n+1} f(x) \| \\ &\le \max\{ \| 2f(2x) + (n-1)(n-2)(n-3) f(0) - 2^{n+1} f(x) \| \\ &\quad , \| (n-1)(n-2)(n-3) f(0) \| \} \\ &\le \max\{ \zeta_n(x,0), a_n \zeta_n(0,0) \}. \end{aligned} \tag{11}$$

for all $x \in G$. So

$$\left\| \frac{f(2x)}{2^n} - f(x) \right\| \le \frac{1}{|2|^{n+1}} \max\{ \zeta_n(x,0), a_n \zeta_n(0,0) \} \tag{12}$$

for all $x \in G$. Replacing x by $2^m x$ and dividing both sides by $|2|^{mn}$ in (12), we get

$$\left\| \frac{f(2^{m+1} x)}{2^{(m+1)n}} - \frac{f(2^m x)}{2^{mn}} \right\| \le \frac{1}{|2|^{(m+1)n+1}} \max\{ \zeta_n(2^m x, 0), a_n \zeta_n(0,0) \} \tag{13}$$

for all $x \in G$. It follows from (5) and (13) that sequence $\left\{ \frac{f(2^m x)}{2^{mn}} \right\}_{m \ge 1}$ is a Cauchy sequence in complete non-Archimedean space X, and so is convergent. Set

$$\vartheta(x) := \lim_{m\to\infty} \frac{f(2^m x)}{2^{mn}}$$

Using induction on m, one can easily see that

$$\left\|\frac{f(2^m x)}{2^{mn}} - f(x)\right\| \le \max\left\{\frac{1}{|2|^{(k+1)n+1}}\zeta_n(2^k x,0), \frac{1}{|2|^{(k+1)n+1}} a_n\zeta_n(0,0);\ 0\le k\le m\right\}. \quad (14)$$

By taking m to approach infinity in (14) and using (6) one obtains (8). To show $\vartheta(x)$ satisfies (4), replace x and y by $2^m x$ and $2^m y$, respectively, in (7) and divide by 2^{mn}, we obtain

$$\begin{aligned}\frac{1}{|2|^{mn}}\| f(2^{m+1}x+2^m y) + f(2^{m+1}x-2^m y) + (n-1)(n-2)(n-3)f(2^m y) \\ - 2^{n-2}\left[f(2^m x+2^m y)+f(2^m x-2^m y)+6f(2^m x)\right]\| \\ \le \frac{1}{|2|^{mn}}\zeta_n(2^m x, 2^m y)\end{aligned}$$

for all $x, y\in G$ and all $m\in N$. Taking the limit as $m\to\infty$, we find that $\vartheta(x)$ satisfies (4) for all $x, y\in G$.

To prove the uniqueness of the mapping $\vartheta(x)$. Let η be another mapping satisfying (8), then for $x\in G$, we get

$$\begin{aligned}\|\vartheta(x)-\eta(x)\|_X &= \lim_{j\to\infty}|2|^{-jn}\|\vartheta(2^j x)-\eta(2^j x)\|_X \\ &\le \lim_{j\to\infty}|2|^{-jn}\max\left\{\|\vartheta(2^j x)-f(2^j x)\|, \|\eta(2^j x)-f(2^j x)\|\right\} \\ &\le \frac{1}{|2|}\lim_{j\to\infty}\lim_{m\to\infty}\left\{\frac{\zeta_n(2^k x,0)}{|2|^{kn}}, \frac{a_n\zeta_n(0,0)}{|2|^{kn}};\ j\le k< m+j\right\} \\ &= 0.\end{aligned}$$

Therefore, $\vartheta = \eta$. This completes the proof.

Corollary 1. For each $n = 1,2,3,4,$ let $\eta:[0,\infty)\to[0,\infty)$ be a function satisfying

$$\eta(|2|t)\le\eta(2)\,\eta(t)\,(t\ge 0),\quad \eta(|2|)<|2|^n.$$

Let $\delta>0$ and $f: G\to X$ be a mapping satisfying

$$\|\Omega_f^n(x,y)\|_X \le \delta(\eta(|x|)+\eta(|y|))$$

for all $x, y\in G$. Then there exists a unique mapping $\vartheta: G\to X$ such that

$$\|f(x)-\vartheta(x)\|_X \le \frac{\delta\eta(|x|)}{|2|}$$

Proof: Defining $\zeta_n: G^2\to[0,\infty)$ by $\zeta_n(x,y) := \delta(\eta(|x|)+\eta(|y|))$, since $|2|^{-n}\eta(|2|)<1$, then we obtain that for all $x, y\in G$

$$\lim_{m\to\infty}\frac{\zeta_n(2^m x, 2^m y)}{|2|^{mn}} \le \lim_{m\to\infty}\left(\frac{\eta(|2|)}{|2|^n}\right)^m \zeta_n(x,y) = 0$$

Also,

$$\begin{aligned}\Omega(x) &= \lim_{m\to\infty}\max\left\{\frac{\zeta_n(2^k x,0)}{|2|^{kn}}, \frac{a_n\zeta_n(0,0)}{|2|^{kn}};\ 0\le k\le m\right\} \\ &= \max\{\zeta_n(x,0), a_n\zeta_n(0,0)\}\end{aligned}$$

and,

$$\lim_{j\to\infty}\lim_{m\to\infty}\left\{\frac{\zeta_n(2^k x,0)}{|2|^{kn}}, \frac{a_n\zeta_n(0,0)}{|2|^{kn}};\ j\le k<m+j\right\} = 0.$$

Applying Theorem 3, the desired result is obtained.

Theorem 4. For each $n = 1,2,3,4$, let $\varsigma_n: G^2\to[0,+\infty)$ be a function such that

$$\lim_{m\to\infty} 2^{mn}\varsigma_n\left(\frac{x}{2^m}, \frac{y}{2^m}\right) = 0 \qquad (15)$$

for all $x, y\in G$. Let for each $x\in G$, the limit

$$\Omega(x) = \lim_{m\to\infty}\max\left\{|2|^{kn}\zeta_n\left(\frac{x}{2^{k+1}},0\right), |2|^{kn}a_n\zeta_n(0,0); 0\le k<m\right\} \quad (16)$$

exists. Suppose that $f: G\to X$ be a mapping satisfying the inequality

$$\|\Omega_f^n(x,y)\| \le \zeta_n(x,y) \qquad (17)$$

for all $x, y\in G$. Then the limit

$$\vartheta(x) := \lim_{m\to\infty} 2^{mn} f\left(\frac{x}{2^m}\right)$$

exists for all $x\in G$ and $\vartheta(x): G\to X$ is a mapping satisfying

$$\|f(x)-\vartheta(x)\| \le \frac{1}{|2|}\Omega(x) \qquad (18)$$

for all $x\in G$. Moreover, if

$$\lim_{j\to\infty}\lim_{m\to\infty}\max\left\{|2|^{kn}\zeta_n\left(\frac{x}{2^{k+1}},0\right), |2|^{kn}a_n\zeta_n(0,0);\ j\le k<m+j\right\} = 0$$

Then $\vartheta(x)$ is the unique mapping satisfying (18).

Proof: By (12), we have

$$\|f(2x)-2^n f(x)\| \le \frac{1}{|2|}\max\{\zeta_n(x,0), a_n\zeta_n(0,0)\} \quad (19)$$

Replacing x by $\frac{x}{2^m}$ in (19), we obtain

$$\left\| 2^{(m-1)n} f\left(\frac{x}{2^{m-1}}\right) - 2^{mn} f\left(\frac{x}{2^m}\right) \right\|_x \le |2|^{n(m-1)-1} \max\left\{\zeta_n\left(\frac{x}{2^m},0\right), a_n\zeta_n(0,0)\right\} \quad (20)$$

for all $x \in G$ and all non-negative integer m. It follows from (15) and (20) that the sequence $\left\{2^{mn} f\left(\frac{x}{2^m}\right)\right\}_{m=1}^{\infty}$ is a Cauchy in X for all $x \in G$. Since X is complete, the sequence $\left\{2^{mn} f\left(\frac{x}{2^m}\right)\right\}_{m=1}^{\infty}$ converges for all $x \in G$. On the other hand, it follows from (20) that

$$\begin{aligned}\left\|2^{pn} f\left(\frac{x}{2^p}\right) - 2^{qn} f\left(\frac{x}{2^q}\right)\right\| &= \left\|\sum_{k=p}^{q-1} 2^{(k+1)n} f\left(\frac{x}{2^{k+1}}\right) - 2^{kn} f\left(\frac{x}{2^k}\right)\right\| \\ &\le \max\left\{\left\|2^{(k+1)n} f\left(\frac{x}{2^{k+1}}\right) - 2^{kn} f\left(\frac{x}{2^k}\right)\right\|; p \le k < q-1\right\} \\ &\le \frac{1}{|2|}\max\left\{|2|^{kn}\zeta_n\left(\frac{x}{2^{k+1}},0\right), |2|^{kn} a_n\zeta_n(0,0); p \le k < q\right\},\end{aligned}$$

for all $x \in G$ and all non–negative integers p,q with $q > p \ge 0$. Letting $p = 0$ and passing the limit $q \to \infty$ in the last inequality and using (16), we obtain (18).

The rest of the proof is similar to the proof of Theorem 3.

Corollary 2. For each $n = 1,2,3,4,$ let $\eta : [0,\infty) \to [0,\infty)$ be a function satisfying

$$\eta\left(|2|^{-1} t\right) \le \eta\left(|2|^{-1}\right)\eta(t) \ (t \ge 0), \quad \eta\left(|2|^{-1}\right) < |2|^{-n}$$

Let $\delta > 0$ and $f : G \to X$ is a mapping satisfying

$$\|\Omega_f^n(x,y)\|_X \le \delta\left(\mu(|x|) + \mu(|y|)\right)$$

for all $x, y \in G$. Then there is a unique mapping $\vartheta : G \to X$ such that

$$\|f(x) - \vartheta(x)\|_x \le \frac{\delta\eta(|2|)}{|2|^{n+1}}$$

Proof. Defining $\zeta_n : G^2 \to [0,\infty)$ by $\zeta_n(x,y) := \delta\left(\mu(|x|) + \mu(|y|)\right)$, then we obtain

$$\lim_{m\to\infty} 2^{mn}\zeta_n\left(\frac{x}{2^m}, \frac{y}{2^m}\right) = 0.$$

Also,

$$\begin{aligned}\Omega(x) &= \lim_{m\to\infty} \max\left\{|2|^{kn}\zeta_n\left(\frac{x}{2^{k+1}},0\right), |2|^{kn} a_n\zeta_n(0,0); 0 \le k < m\right\} \\ &= \varsigma_n\left(\frac{x}{2},0\right) \\ &\le |2|^{-n}\delta\mu(|x|)\end{aligned}$$

And

$$\lim_{j\to\infty}\lim_{m\to\infty} \max\left\{|2|^{kn}\zeta_n\left(\frac{x}{2^{k+1}},0\right), |2^{kn}| a_n\zeta_n(0,0); j \le k < m+j\right\} = 0.$$

4. Non- Archimedea stability of functional equation (4): fixed point method

Throughout this section, assume that X is a non-Archimedean normed vector space and that Y is a non- Archimedean Banach space. In the rest of the present paper, let $|2| \ne 1$.

Theorem 5. For $n = 1,2,3,4,$ $\zeta_n : X \to [0,\infty)$ be a function such that there exists an $L < 1$ with

$$\zeta_n(2x, 2y) \le |2|^n L\zeta_n(x,y) \quad (21)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|\Omega_f^n(x,y)\| \le \zeta_n(x,y) \quad (22)$$

for all $x, y \in X$. Then there is a unique mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\| \le \frac{\max\{\zeta_n(x,0), a_n\zeta_n(0,0)\}}{|2|^{n+1}(1-L)} \quad (23)$$

Proof: By (12), we have

$$\|2f(2x) - 2^{n+1} f(x)\| \le \max\{\zeta_n(x,0), a_n\zeta_n(0,0)\}. \quad (24)$$

for all $x \in X$. Consider the set

$$S := \{g : X \to Y\}$$

and the generalized metric d in S defined by

$$d(f,g) = \inf\left\{\mu \in R^+ : \|g(x) - h(x)\| \le \mu \max\{\zeta_n(x,0), a_n\zeta_n(0,0)\}, \forall x \in X\right\},$$

where $\inf\varphi = +\infty$. It is easy to show that (S,d) is complete. Now, we consider a linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2^n} h(2x)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon \max\{\zeta_n(x,0), a_n \zeta_n(0,0\}$$

for all $x \in X$. So

$$\|Jg(x) - Jh(x)\| = \left\| \frac{1}{2^n} g(2x) - \frac{1}{2^n} h(2x) \right\|$$
$$\le \frac{\varepsilon}{|2|^n} \max\{\zeta_n(2x,0), a_n \zeta_n(0,0)\}$$
$$\le \frac{1}{|2|^n} \varepsilon |2|^n L \max\{\zeta_n(x,0), a_n \zeta_n(0,0)\}$$

for $x \in X$. Thus $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$, this means that $d(Jg, Jh) \le L d(g,h)$ f or all $g, h \in S$. It follows from (24) that $d(f, Jf) \le \frac{1}{|2|^{n+1}}$.

By Theorem 2, there exists a mapping $C : X \to Y$ satisfying the following :

(i) C is a fixed point of J, that is, for all $x \in X$,

$$C(2x) = 2^n C(x) \tag{25}$$

(ii) the mapping C is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that C is a unique mapping satisfying (25) such that there exists $\mu \in (0,\infty)$ satisfying $\|f(x) - C(x)\| \le \mu \max\{\zeta_n(x,0), a_n \zeta_n(0,0)\}$, for all $x \in X$.

(iii) $d(J^m f, C) \to 0$ as $m \to \infty$. This implies the equality, $\lim_{m\to\infty} \frac{f(2^m x)}{2^{mn}} = C(x)$, for all $x \in X$.

(iv) $d(f, C) \le \frac{d(f, Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f, C) \le \frac{1}{|2|^{n+1}(1-L)}$.

This implies that the inequality (23) holds.

Corollary 3. Let $\theta \ge 0$ and p be a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying

$$\|\Omega_f^n(x,y)\| \le \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. Then, the limit $C(x) = \lim_{m\to\infty} \frac{f(2^m x)}{2^{mn}}$ exists for all $x \in X$ and $C : X \to Y$ is a unique mapping such that

$$\|f(x) - C(x)\| \le \frac{|2|^{np} \theta \|x\|^p}{|2|^{n+1}\left(|2|^{np} - |2|^n\right)}$$

for all $x \in X$.

Proof: The proof follows from Theorem 5 by taking $\zeta_n(x,y) = \theta\left(\|x\|^p + \|y\|^p\right)$, for all $x, y \in X$. In fact, if we choose $L = \frac{|2|^n}{|2|^{np}}$ we get the desired result.

Theorem 6. For $n = 1,2,3,4$, let $\zeta_n : X \to [0,\infty)$ be a function such that there exists an $L < 1$ with

$$\zeta_n(x,y) \le \frac{L}{|2|^n} \zeta_n(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\|\Omega_f^n(x,y)\| \le \zeta_n(x,y)$$

for all $x, y \in X$. Then there is a unique mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\| \le \frac{L \max\{\zeta_n(x,0), a_n \zeta_n(0,0)\}}{|2|^{n+1}(1-L)} \tag{26}$$

Proof: By (11), we have

$$\left\| f(x) - 2^n f\left(\frac{x}{2}\right) \right\| \le \frac{1}{|2|} \max\left\{ \zeta_n\left(\frac{x}{2}, 0\right), a_n \zeta_n(0,0) \right\} \tag{27}$$

for all $x \in X$. Let (S,d) be the generalized metric space defined as in the proof of Theorem 5, we consider a linear mapping $J : S \to S$ such that $Jh(x) := 2^n h\left(\frac{x}{2}\right)$ for all $x \in X$. Let $g, h \in S$ be such that $d(g,h) = \varepsilon$. Then $\|g(x) - h(x)\| \le \varepsilon \max\{\zeta_n(x,0), a_n \zeta_n(0,0\}$ for all $x \in X$. So

$$\|Jg(x) - Jh(x)\| = \left\|2^n g\left(\frac{x}{2}\right) - 2^n h\left(\frac{x}{2}\right)\right\|$$
$$\leq |2|^n \varepsilon \max\left\{\zeta_n\left(\frac{x}{2},0\right), a_n\zeta_n(0,0)\right\}$$
$$\leq |2|^n \varepsilon \frac{L}{|2|^n} \max\left\{\zeta_n(x,0), a_n\zeta_n(0,0)\right\}$$

for all $x \in X$. Thus $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$, this means that $d(Jg, Jh) \leq Ld(g,h)$ for all $g,h \in S$. It follows from (27) that $d(f, Jf) \leq \frac{L}{|2|^{n+1}}$.

By Theorem 2, there exists a mapping $C: X \to Y$ satisfying the following:

(a) C is a fixed point of J, that is

$$C\left(\frac{x}{2}\right) = \frac{1}{2^n}C(x) \tag{28}$$

for all $x \in X$.

(b) The mapping C is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies C is a unique mapping satisfying (28) such that there exists $\mu \in (0,\infty)$ satisfying $\|f(x) - C(x)\| \leq \mu \max\{\zeta_n(x,0), a_n\zeta_n(0,0)\}$, for all $x \in X$.

(c) $d(J^m f, C) \to 0$ as $m \to \infty$, this implies the equality $\lim_{n\to\infty} 2^{mn} f\left(\frac{x}{2^m}\right) = C(x)$ for all $x \in X$.

(d) $d(f, C) \leq \frac{d(f, Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f, C) \leq \frac{L}{|2|^{n+1}(1-L)}$.

This implies that the inequality (26) holds.

The rest of the proof is similar to the proof of Theorem 5.

Corollary 4. Let $\theta \geq 0$ and p be a real number with $p > 1$. Let $f: X \to Y$ be a mapping satisfying

$$\left\|\Omega_f^n(x,y)\right\| \leq \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. Then, the limit $C(x) = \lim_{m\to\infty} 2^{mn} f\left(\frac{x}{2^m}\right)$ exists for all $x \in X$, and $C: X \to Y$ is a mapping such that

$$\|f(x) - C(x)\| \leq \frac{|2|^{np}\,\theta\|x\|^p}{|2|^{n+1}\left(|2|^n - |2|^{np}\right)}$$

for all $x \in X$.

Proof: The proof follows from Theorem 6 by taking $\zeta_n(x,y) = \theta\left(\|x\|^p + \|y\|^p\right)$ for all $x, y \in X$. In fact, if we choose $L = \frac{|2|^{np}}{|2|^n}$, we get the desired result.

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