

An Efficient Procedure for Computing an Optimal (R, Q) Policy in Continuous Review Systems with Poisson Demands and Constant Lead Time

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Abstract. *In this paper, a continuous review inventory system is considered in which an order in a batch of size Q is placed immediately after the inventory position reaches R . Transportation time is constant and demands are assumed to be generated by a stationary Poisson process with one unit demand at a time. Demands not covered immediately from the inventory are backordered. In a recent paper, the exact evaluation of batch-ordering policies for two-level inventory systems was derived. This evaluation is based on a recursive procedure for determining the exact policy costs in case of one-for-one replenishment policies. In this paper, we show how this result can be applied to find the optimal solution of a (R, Q) policy. To obtain the optimal policy for this system, considering a one-for-one policy, we will first solve the base stock model by setting the inventory position at the supplier to a certain value. By considering ordering cost, we next derive the cost function of the defined (R, Q) model and find the optimal solution for the exact value of the expected system costs using a search method. In demonstrating the applicability of the proposed method, we resort to solving an example.*

Keywords: *Inventory; Continuous review; (R, Q) model; Base stock model; Poisson demand; Optimal solution; Backordered demand; Constant lead time.*

INTRODUCTION

Stochastic inventory model have received considerable attention in inventory literature. We consider one of the most common practical stochastic inventory control problems, known as the continuous (R, Q) model. In this system, a batch of size Q is ordered when the inventory position declines to R . The model and some of its variants have been analyzed by several authors [1,2].

The basic assumptions of the classical (R, Q) can be found in Hadley and Whitin [3]. The optimal solution of this problem and other related problems can be determined by various techniques [4,5]. In practice, however, it is most common to use a simple two-step approximation. In the first step, the batch quantity Q is determined in a deterministic model.

One such approach is to determine the order quantity according to the standard EOQ model. Axsäter [6] and Gallego [7] have derived bounds for approximation errors when using such techniques.

In a (R, Q) inventory system with backordering, there are several approximations to the average inventory level. Lau and Lau [8] presented a comparison of different methods for estimating the average inventory level in this type of system.

Axsäter [9] considered a (R, Q) policy with constant lead time. The demand during lead-time demand is assumed to be normally distributed. There are standard ordering and holding costs as well as a so-called fill rate constraint. The problem is to determine the reorder point R and the order quantity Q so that the total expected costs are minimized. He suggested another equally two-step approach and both R and Q were assumed to be continuous variables.

Axsäter [10] also considered a warehouse facing a compound Poisson customer demand. Normally, the warehouse replenishes from an outside supplier, according to a continuous review reorder point policy.

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However, it is also possible to use emergency orders. Such orders incur additional costs, but have a much shorter lead time. He considered standard holding and backorder costs as well as ordering costs. A heuristic decision rule for triggering emergency orders was suggested.

Modularization and customization have made enterprises face multi-item inventory problems and the interaction between those items. A powerful and affordable information technology system can make the continuous review inventory policy more convenient, efficient and effective. Others have introduced different aspects of cost into their models such as design problems or flexibility in the assembly sequence. The aforementioned considerations make models more complex and the closed form solutions cannot be obtained. Thus, iterative algorithms must be used to find a near optimal solution. Several studies have presented some efficient solution procedures to obtain the approximating decision rules for such problems [11,12].

Wang and Hu [13] developed a (R, Q) model to find the optimal lot size and reorder point for a multi-item inventory with interactions between necessary and optional components. In order to accurately approximate costs, the service cost is introduced and defined as a proportion of the service level. In addition, the service and purchasing costs are taken simultaneously and are treated as a budget constraint for executives to consider because the firm's strategy could influence the choice of service level. They formulated the proposed model as a nonlinear optimization problem, when the service level is nonlinear.

A recent (R, Q) model was presented by Lau and Lau [14]. When an order for an out-of-stock inventory item is received, sometimes a fixed cost independent of the order size is incurred to expedite the order. They showed that the current standard "textbook formulation" for this situation contains a number of conceptual flaws. An improved formulation is presented, followed by explanations of computation procedures and numerical illustrations.

There are several studies in which the (R, Q) policy is applied in multi-echelon inventory systems. Axsäter [15] considered an inventory system with one warehouse and some identical retailers. Lead times are constant and the retailers face independent Poisson demand. When the retailers need to replenish their stocks, they order batches from the supplier and, in the same way, the warehouse replenishes its stock by ordering batches from an outside source. Solving the mentioned inventory system, Axsäter extended his previous results in [16], where replenishments are a one-for-one policy. He provided simple recursive procedures for determining the holding and shortage costs of different one-for-one policies.

Here, we consider the continuous review inven-

tory system that is controlled using a (R, Q) policy. Demands are assumed to be generated by a stationary Poisson process with one unit demand at a time. Unfilled demand is backordered and the cost for a backorder is proportional to the delay until delivery takes place. The transportation time is constant.

As mentioned before, Axsäter [15,16] derived the procedures for determining the best solution for a one-for-one and a batch-ordering policy in a two-level inventory system. Setting the inventory position at the supplier to a certain value, we show how these procedures can be used for finding the optimal solution of base stock and (R, Q) policies.

Our approach uses an inventory cost function that reflects costs incurred on an average unit. The relative advantage of the approach in this paper is that it focuses directly on evaluating the average costs associated with a stockage policy. Earlier approaches focus on characterizing the steady-state behavior of the inventory levels and then use the steady state distribution (or an approximation thereof) to determine the average costs associated with the stockage policy. Axsäter [16] mentions that this approach is more efficient and direct at finding the optimal stock policy for the assumed (traditional) cost function; it appears to be the only available approach when the cost is given by a nonlinear function of either the delays experienced by the customer or the unit's storage time at each of the facilities.

The rest of the paper is organized as follows: In the next section, the one-for-one replenishment policy and its assumptions are described and a problem formulation is presented in detail. Then, we show how the result of the one-for-one replenishment policy in a two-level inventory system can be used for the exact evaluation of the base stock policy. Following that, the batch-ordering policy in a two-level inventory system is extended by considering order costs for both retailer and supplier. Setting the inventory position at the supplier to a certain value, a search algorithm for finding the optimal solution of a (R, Q) policy is presented. Then, for demonstrating the applicability of the proposed method, we resort to solving an example. Finally, some remarks concerning possible extension are made.

DESCRIPTION AND PROBLEM FORMULATION

Consider an inventory system that consists of one supplier and one retailer, as shown in Figure 1. The retailer faces stationary and independent Poisson demand with one-unit demand at a time. Unfilled demand is backordered and the shortage cost is considered just for the retailer. The retailer and the supplier carry

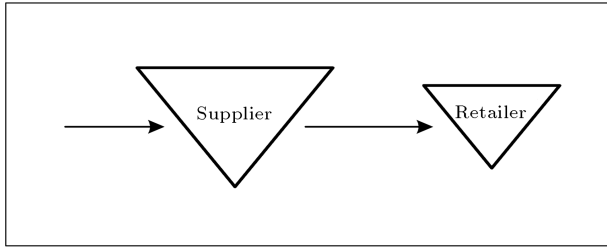


Figure 1. Inventory system.

inventory and replenish their stock according to a one-for-one policy, i.e. when a demand occurs, one unit is immediately ordered from the supplier and the supplier orders one unit at the same time. It means essentially that it is assumed that ordering costs are low and can be disregarded. Delayed demands and delayed orders are satisfied on a first come, first serve basis. The transportation time from the supplier to the retailer and the replenishment lead time from the outside source to the supplier are constant (i.e. the outside source has ample capacity).

A one-for-one replenishment policy is completely characterized by determination of the inventory positions at the retailer and at the supplier.

Considering the one-for-one policy, we will solve the base stock model by setting the inventory position at the supplier to a certain value and then finding the optimal solution for this model.

DERIVING THE COST FUNCTION AND FINDING THE OPTIMAL SOLUTION FOR THE BASE STOCK MODEL

In the base stock model, when a demand occurs at the retailer, one unit is immediately ordered from the outside source. We denote the base stock policy by $(R_B, 1)$, i.e. when inventory position declines to R_B one unit is ordered. This base stock policy is completely characterized by determination of the retailer's reorder point. The ordering cost per time unit in all base stock policies with one unit demand at a time is constant, and thus is not considered in the analysis. In this section, we try to find the optimal value of R_B as a decision variable.

We convert the assumed one-for-one model to the defined base stock model (Figure 2). To find the optimal value of R_B , our approach uses the optimal retailer inventory position in the inventory system with the one-for-one policy, when setting the value of the inventory position at the supplier to 0.

Lemma 1.

A one-for-one replenishment system with $S, S_0 = 0, L$ and L_0 is the same as a base stock system with lead time $L + L_0$ and $R_B = S - 1$.

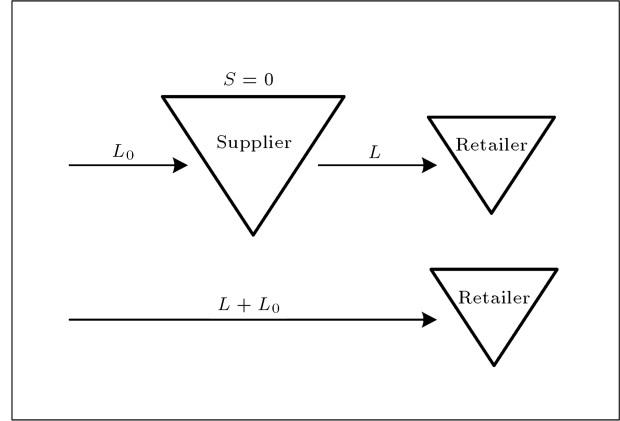


Figure 2. The equivalent systems based on Lemma 1.

Proof

When a demand occurs at the retailer, one unit is immediately ordered from the supplier and the supplier orders one unit at the same time. When demand occurs, the supplier is empty; because the inventory position at the supplier is zero, the supplier receives the ordered unit after L_0 time units (all the outstanding orders are assigned before they arrive at the supplier and the supplier must order new one to assign to the new one ordered), and it is received by the retailer L time units away. Therefore, an order placed by the retailer arrives after $L_0 + L$ time units.

Based on Lemma 1, if S^* minimizes the cost function of the assumed one-for-one replenishment system, $R_B^* = S^* - 1$ will minimize the cost function of the base stock system. It is easy to conclude that we can disregard policies with a negative inventory position for the retailer in the assumed one-for-one ordering system when looking for the optimal solution. Therefore, confine ourselves to the case where $S \geq 0$ for the base stock model.

To evaluate the total holding and shortage costs per time unit when applying one-for-one ordering policies (i.e. $c(S_0, S)$), we use the method introduced by Axsäter [15,16] (see Appendix).

The total cost of the assumed one-for-one replenishment system can be calculated by Equation A13:

$$c(0, S) = \lambda(\Pi^S(0) + \gamma(0)). \tag{1}$$

Considering Equations A4 and A8, it is easy to simplify Expression 1 as:

$$c(0, S) = \lambda\pi^S(L_0). \tag{2}$$

It means that if S^* minimizes $\pi_S(L_0)$, it will minimize $c(0, S)$. In order to optimize $\pi^S(L_0)$, we need to determine the S that minimizes $\pi^S(L_0)$.

Based on Equations A5 and A6, we can derive:

$$\pi^1(L_0) - \pi^0(L_0) = e^{-\lambda(L+L_0)} \frac{h + \beta}{\lambda} - \frac{\beta}{\lambda}, \quad (3)$$

$$\pi^{S+1}(L_0) - \pi^S(L_0) = \pi^S(L_0) - \pi^{S-1}(L_0) + e^{-\lambda(L+L_0)} \frac{h + \beta (L + L_0)^S \lambda^S}{S!}, \quad S > 0. \quad (4)$$

Using Equation 4, it is easy to conclude that $\pi^S(L_0)$ is convex in S . Furthermore, we obtain the following expression:

$$\begin{aligned} \pi^{S+1}(L_0) - \pi^S(L_0) &= \frac{h + \beta}{\lambda} \sum_{k=0}^S e^{-\lambda(L+L_0)} \\ &\quad \frac{(L + L_0)^k \lambda^k}{k!} - \frac{\beta}{\lambda} \\ &= \frac{h + \beta}{\lambda} P(S, \lambda(L + L_0)) - \frac{\beta}{\lambda}, \end{aligned} \quad (5)$$

$S \geq 0.$

Based on convexity specifications, the optimal value of S that minimizes $\pi^S(L_0)$ will be the minimum value of S that is valid for:

$$P(S, \lambda(L + L_0)) \geq \frac{\beta}{\beta + h}. \quad (6)$$

Therefore, the optimal value of R_B in a base stock system with lead time $L + L_0$ will be $S^* - 1$ where S^* minimizes the cost function of the assumed one-for-one replenishment system.

DERIVING THE COST FUNCTION AND FINDING THE OPTIMAL SOLUTION FOR THE LOT SIZE-REORDER POINT MODEL

In this section, we try to find the optimal value of the reorder point and lot size as decision variables. In the first part of this section, the cost function for a lot size-reorder point model is developed and extended by considering the order cost for the retailer. In the second part of this section, for a given lot size, the optimal value of the retailer's reorder point will be found, and finally in the last part of this section, a search method to find the optimal solution of the lot size-reorder point model will be provided. These three sub sections are presented as follows:

Cost Function for the Lot Size-Reorder Point Model

The cost function for a serial system that consists of a retailer and a supplier is developed in Axsäter [15].

In this system, both facilities apply a lot size-reorder point policy, where R_w and Q_w are in units of Q . The average cost per time unit, that is valid for $R_w \geq -1$, is determined by averaging over the individual units in a warehouse batch:

$$C = \frac{1}{Q_w Q} \sum_{j=R_w+1}^{R_w+Q_w} \sum_{k=R+1}^{R+Q} c(jQ, k), \quad R_w \geq -1. \quad (7)$$

In developing the model, Axsäter [15] made a few assumptions. He assumed that the order lot sizes at the retailer and the supplier are given and fixed, therefore the order costs have been eliminated from the cost function (Equation 7). It means that the ratio of order cost to unit holding cost at the supplier and the ratio of order cost to unit holding and shortage costs at the retailer are negligible. In cases where shipping and handling are significant, this assumption is violated.

We extend the model by considering the order costs of both the retailer and the supplier (Figure 3). By considering the rates of orders, the expected total holding and shortage and order costs per time unit can be written as:

$$\begin{aligned} C' &= \frac{\lambda A_0}{Q_w Q} + \frac{\lambda A}{Q} + \\ &\quad \frac{1}{Q_w Q} \sum_{j=R_w+1}^{R_w+Q_w} \sum_{k=R+1}^{R+Q} c(jQ, k), \quad R_w \geq -1. \end{aligned} \quad (8)$$

Lemma 2

A serial system that consists of one retailer and one supplier and the retailer applies the (R, Q) policy and the supplier applies (R_w, Q_w) where R_w and Q_w are in units of Q with $R_w = -1, Q_w = 1, A_0 = 0, L$ and L_0 is the same as a lot size-reorder point system with lead time $L + L_0$ where the retailer applies (R, Q) .

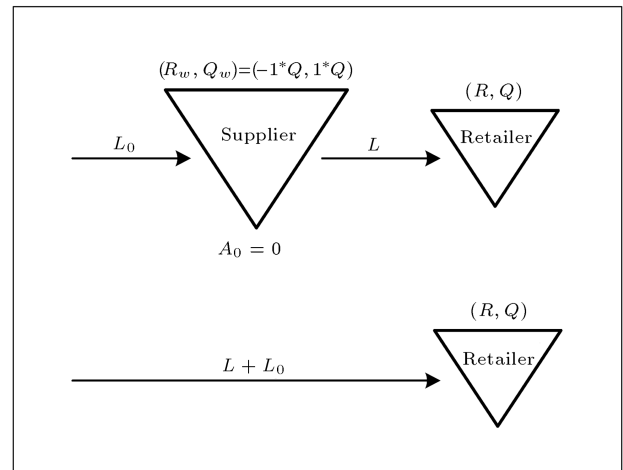


Figure 3. The equivalent systems based on Lemma 2.

Proof

When the inventory position at the retailer reaches R , one order (of size Q) is immediately ordered from the supplier and the supplier orders one order (in the same size) at the same time. When demand occurs, the supplier is empty because the inventory position at the supplier is zero, the supplier receives the ordered unit after L_0 time units (all the outstanding orders are assigned before they arrive at the supplier and the supplier must order new ones to assign to the new one ordered), and it is received by the retailer L time units away. Therefore, an order placed by the retailer arrives after $L_0 + L$ time units.

So, if R_w, Q_w and A_0 are replaced with specified values in Lemma 2, we will have:

$$TC(Q, R) = \frac{\lambda A}{Q} + \frac{1}{Q} \sum_{k=R+1}^{R+Q} c(0, k). \tag{9}$$

Optimal Value of the Retailer’s Reorder Point for a Given Lot Size

In order to optimize the lot size-reorder point model for a given lot size, we need to determine the R that minimizes the system costs, according to Equation 9. Note that, for a given Q , we can optimize $TC(Q, R)$ by optimizing $\sum_{k=R+1}^{R+Q} c(0, k)$ with respect to R . Furthermore, it is possible to show that optimal policies for lot size-reorder point models are such that $R \geq -Q$. For $k \geq 0$, it was proven that $c(0, k)$ is convex in k . Using Equations A5 and A6 in the same way, it is easy to show that $c(0, k)$ is convex at all values of k . Therefore, $\sum_{k=R+1}^{R+Q} c(0, k)$ will be convex in R .

The optimal retailer’s reorder point for a given lot size Q is denoted by R_Q^* . As shown in Figure 4, R_Q^* is a point wherein there is a set of Q sequential points

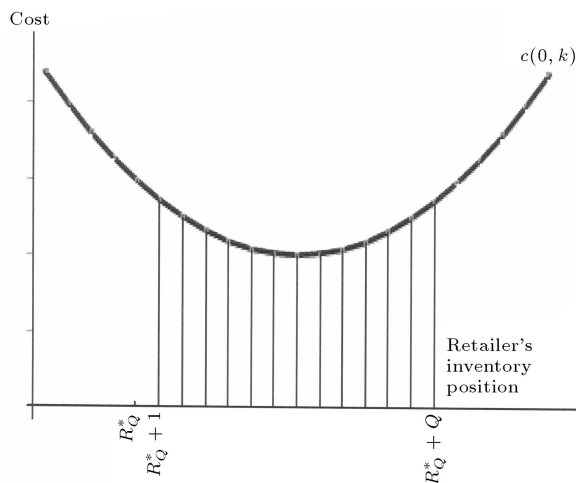


Figure 4. R_Q^* and the optimal set of Q sequential points are exactly after this point.

exactly after this point, where the sum of the values of $c(0, k)$ for all members of this set is the minimum of the sum of the values of $c(0, k)$ for all members of any set of Q sequential points.

For simplicity, we set $R_1^* = R_B^*$. Based on $c(0, k)$ convexity, it is obvious that $R_1^* + 1 \in [R_Q^* + 1, R_Q^* + Q]$, therefore we have:

$$R_1^* - Q + 1 \leq R_Q^* \leq R_1^*. \tag{10}$$

If $R \geq -1$, by using Equation 5, we get:

$$\sum_{k=R+2}^{R+Q+1} c(0, k) - \sum_{k=R+1}^{R+Q} c(0, k) = (h + \beta) \sum_{k=R+1}^{R+Q} P(k, \lambda(L + L_0)) - Q\beta. \tag{11}$$

The optimal value of R that minimizes $\sum_{k=R+1}^{R+Q} c(0, k)$ will be the minimum value of R that is valid for:

$$\sum_{k=R+1}^{R+Q} P(k, \lambda(L + L_0)) \geq \frac{Q\beta}{(h + \beta)}, \quad R \geq -1. \tag{12}$$

If $R < -1$, by using Equations A5 and A6, it can be shown:

$$\sum_{k=R+2}^{R+Q+1} c(0, k) - \sum_{k=R+1}^{R+Q} c(0, k) = (h + \beta) \sum_{k=0}^{R+Q} P(k, \lambda(L + L_0)) - Q\beta. \tag{13}$$

The optimal value of R that minimizes $\sum_{k=R+1}^{R+Q} c(0, k)$ will be the minimum value of R that is valid for:

$$\sum_{k=0}^{R+Q} P(k, \lambda(L + L_0)) \geq \frac{Q\beta}{(h + \beta)}, \quad R < -1. \tag{14}$$

Therefore, for a given order size Q , the optimal value of R that minimizes $TC(Q, R)$ will be found by examining values of R starting from $R_1^* - Q + 1$ to the first value of R that is valid for Equation 14 (for the values less than -1), and then Equation 12 (for the values greater than or equal to -1), respectively.

Furthermore, it may also be of interest to utilize R_{Q-1}^* for finding R_Q^* (Figure 5). Note that, when we have R_{Q-1}^* , we found the set of $Q - 1$ sequential points exactly after this point, where the sum of the values of $c(0, k)$ for the all member of this set, is minimum. When we try to find R_Q^* , we try to add one member (except the selected $Q - 1$ points) to this set, where the values of $c(0, k)$ for the new member is minimum. Based on the convexity of $c(0, k)$, the new member will

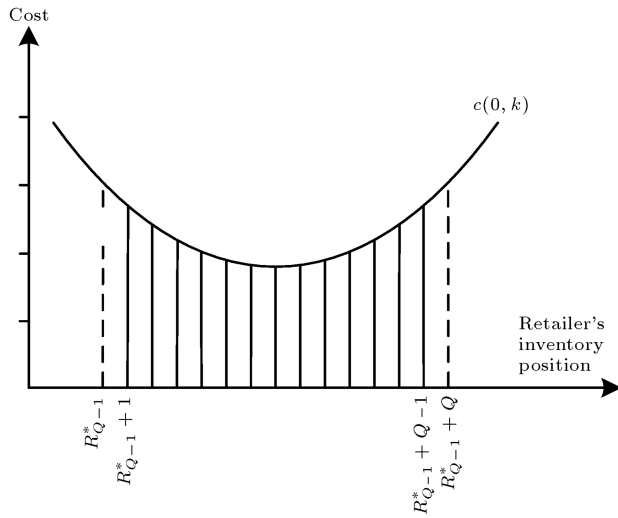


Figure 5. Utilizing R_{Q-1}^* for finding R_Q^* .

be a point exactly before or after the selected $Q - 1$ sequential points. Considering the R_Q^* definition, it can be concluded that:

$$R_Q^* \in [R_{Q-1}^* - 1, R_{Q-1}^*]. \quad (15)$$

If $R_Q^* = R_{Q-1}^*$, it means that $R_{Q-1}^* + Q$ is added to $(Q - 1)$ members that minimize $\sum_{k=R_{Q-1}^*+1}^{R_{Q-1}^*+Q-1} c(0, k)$ and, if $R_Q^* = R_{Q-1}^* - 1$, it means that R_{Q-1}^* is the Q th member of this set. Therefore, we get:

$$\begin{aligned} c(0, R_{Q-1}^*) \leq c(0, R_{Q-1}^* + Q) &\rightarrow R_Q^* = R_{Q-1}^* - 1, \\ c(0, R_{Q-1}^*) > c(0, R_{Q-1}^* + Q) &\rightarrow R_Q^* = R_{Q-1}^*. \end{aligned} \quad (16)$$

A Search Method to Find the Optimal Solution of the Lot Size-Reorder Point Model

To find the optimal solution of the lot size-reorder point model, it is necessary to determine R and Q that minimize the system costs, according to Equation 9. This cost function is comprised of two parts. The first part is a decreasing function on Q which is smaller than or equal to λA , and the second part is an increasing function on Q (where $R = R_Q^*$), that is larger than or equal to $c(0, R_1^* + 1)$.

Let Q^u denote the minimum value of Q that is valid for:

$$\lambda A + c(0, R_1^* + 1) \leq \frac{1}{Q^u} \sum_{k=R_{Q^u+1}^*}^{R_{Q^u}^*+Q^u} c(0, k). \quad (17)$$

Considering the increasing function specifications and Expression 17, for $Q \geq Q^u$, we have:

$$\begin{aligned} TC(1, R_1^*) = \lambda A + c(0, R_1^* + 1) &\leq \frac{1}{Q} \sum_{k=R_{Q^*}^*}^{R_{Q^*}^*+Q} c(0, k) \\ &< \frac{\lambda A}{Q} + \frac{1}{Q} \sum_{k=R_{Q^*}^*}^{R_{Q^*}^*+Q} c(0, k) = TC(Q, R). \end{aligned} \quad (18)$$

It means that Q^u is an upper bound for Q . In the calculations, it may also be of interest to utilize that:

$$\begin{aligned} \pi^S(L_0) &= \frac{h + \beta}{\lambda} \sum_{k=1}^{S-1} P(k, \lambda(L + L_0)) \\ &+ \beta \left(L + L_0 - \frac{S}{\lambda} \right), \quad S > 0, \end{aligned} \quad (19)$$

which follows directly from Equation A5. Meanwhile from Equations 9 and 16, it is possible to conclude that:

$$\begin{aligned} TC(Q + 1, R_{Q+1}^*) &= (Q \times TC(Q, R_Q^*) + \\ &\min(c(0, R_Q^*), c(0, R_Q^* + Q + 1)))/Q + 1). \end{aligned} \quad (20)$$

Finally, we can provide a search method, based on the suggested relations, to find the optimal solution of the lot size-reorder point policies (Figure 6). This method is described by the following flowchart shown in Figure 6.

R_{optimum} and Q_{optimum} are the optimal values of the reorder point and lot size in a (R, Q) policy, respectively.

EXAMPLE

In this section, we will try to show the applicability of our search method by solving a test problem. We have assumed that the value of the parameters A, L, L_0, h, h_0, β and λ are constant and for instance are as:

$$\begin{aligned} A &= 10, \quad L = 1, \quad L_0 = 1, \quad h = 1, \\ h_0 &= 0.1, \quad \beta = 10 \quad \text{and} \quad \lambda = 1. \end{aligned}$$

For a base stock model, the optimal value of R_B with lead time $L + L_0$ will be $S^* - 1$, where S^* is the minimum value of S that is valid for Expression 6, i.e.:

$$P(4, 1 \times (1 + 1)) \geq \frac{10}{10 + 1} \rightarrow S^* = 4 \rightarrow R_B^* = 3.$$

Based on Equation 9, for computing $TC(Q, R)$, we need to calculate the values of $c(0, k)$. Using Equations 2, 19 and A6, these values can be found easily.

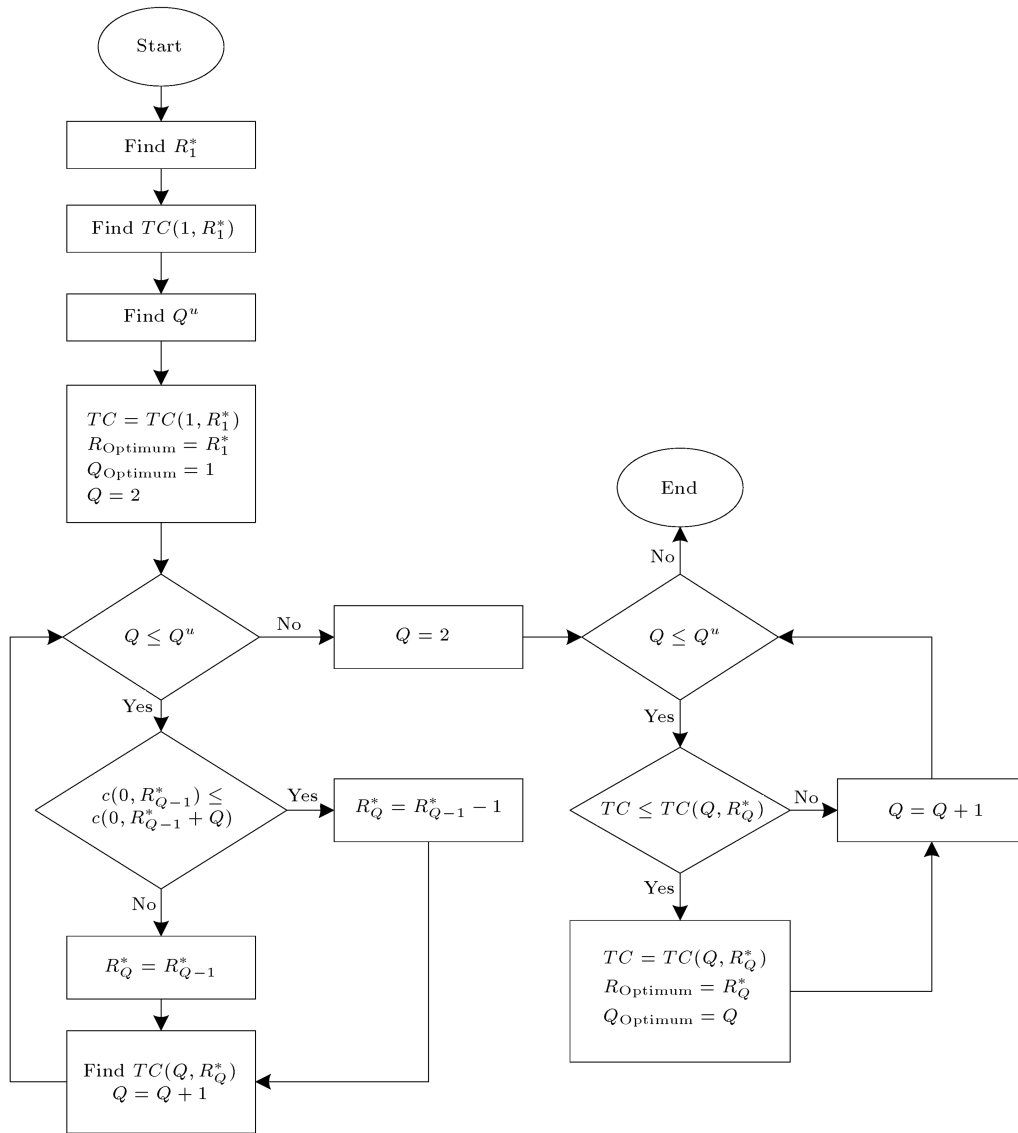


Figure 6. The suggested method to find the optimal solution of lot size-reorder point problems.

As an example, we have:

$$\begin{aligned}
 c(0, -1) &= 30, & c(0, 0) &= 20, & c(0, 1) &= 11.4887, \\
 c(0, 2) &= 5.9548, & c(0, 3) &= 3.3982, & c(0, 4) &= 2.8266, \\
 c(0, 5) &= 3.2474.
 \end{aligned}$$

As said before, $R_1^* = R_B^*$, therefore $R_1^* = 3$. Using Expression 16, we can get R_2^* :

$$c(0, R_1^*) > c(0, R_1^* + 2) \rightarrow R_2^* = R_1^* = 3.$$

In the same way, we can calculate R_Q^* for any value of Q . Using Expression 17, the upper bound for Q can be determined:

$$\begin{aligned}
 TC(1, R_1^*) &= 1 \times 10 + c(0, 4) \\
 &= 12.8266 \leq \frac{1}{28} \sum_{k=-1+1}^{-1+28} c(0, k) \\
 &= 13.0714 \rightarrow Q^u = 28, \quad R_{Q^u}^* = -1.
 \end{aligned}$$

Finally, from Equation 20, we can compute $TC(Q, R_Q^*)$ for any value of Q from 1 to Q^u .

$$\begin{aligned}
 TC(2, R_2^*) &= 8.0370, \\
 TC(3, R_3^*) &= 6.4907, \\
 TC(4, R_4^*) &= 5.8843, \dots, \\
 TC(28, R_{28}^*) &= 13.4286.
 \end{aligned}$$

The minimum value of $TC(Q, R_Q^*)$ will determine the optimal policy in the above example, i.e.:

$$R_{\text{optimum}} = 2, \quad Q_{\text{optimum}} = 5,$$

$$TC(Q_{\text{optimum}}, R_{\text{optimum}}) = 5.7105.$$

CONCLUSION

In this paper, we considered a continuous review inventory system that was controlled using a (R, Q) policy. Demands were assumed to be generated by a stationary Poisson process with one unit demand at a time. Unfilled demand was backordered and the cost for a backorder was proportional to the delay until delivery took place. The transportation time was constant.

First, we solved the base stock model. To derive the cost function of the system, the one-for-one model in a two-level inventory system was converted to this base stock model. For finding the optimal value of the retailer's reorder point in the base stock model, our approach used the optimal retailer inventory position in the inventory system with the one-for-one policy, when setting the value of the inventory position at the supplier to 0.

Then, the batch-ordering policy in a two-level inventory system was extended by considering two order costs for retailer and supplier. Setting the inventory position at the supplier to a certain value, a search algorithm for finding the optimal solution of a (R, Q) policy was provided.

Earlier approaches focus on characterizing the steady-state behavior of the inventory levels of a stockage policy and then use the steady state distribution (or an approximation thereof) to determine the average costs associated with the stockage policy. The proposed approach is more efficient and direct at finding the optimal stockage policy for the traditional cost function. For this reason, the proposed approach can be used when the cost is given by a nonlinear function of either the delays experienced by the customer, or the unit's storage time at the retailer.

NOMENCLATURE

R_B	retailer's reorder point in an inventory system with base stock policy,
S_0	supplier inventory position in an inventory system with a one-for-one ordering policy,
S	retailer inventory position in an inventory system with a one-for-one ordering policy,
L	transportation time from the supplier to the retailer,

L_0	transportation time from the outside source to the supplier,
λ	demand intensity at the retailer
h	holding cost per unit per time unit at the retailer,
h_0	holding cost per unit per time unit at the supplier,
β	shortage cost per unit per time unit at the retailer,
R	the retailer reorder point,
Q	the retailer batch size,
R_w	the warehouse reorder point (in units of retailer batches),
Q_w	the warehouse batch size (in units of retailer batches),
A	fixed order cost at the retailer,
A_0	fixed order cost at the supplier,
$\Pi^S(S_0)$	the expected retailer inventory carrying and shortage costs, which is incurred to fill a unit of demand at the retailer when applying one-for-one ordering policies with S_0 and S as the supplier and the retailer inventory positions, respectively,
$\gamma(S_0)$	the average warehouse holding cost per unit when applying one-for-one ordering policies with S_0 as the supplier inventory positions,
$\pi^S(L_0)$	the expected retailer inventory carrying and shortage costs which is incurred to fill a unit of demand at the retailer when applying one-for-one ordering policies with S as the retailer inventory positions and L_0 as the delay encountered at the supplier level,
$c(S_0, S)$	the total holding and shortage costs per time unit when applying one-for-one ordering policies with S_0 and S as the supplier and the retailer inventory positions, respectively,
$TC(Q, R)$	the total cost per time unit for the retailer when the retailer applies a (R, Q) policy with lead time $L + L_0$,
$p(u, \lambda)$	probability mass function of Poisson with mean λ ,
$P(u, \lambda)$	cumulative probability distribution function of Poisson with mean λ .

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APPENDIX

Evaluation of the One-for-One Ordering Policies

This Appendix is a summary of Axsäter [15,16]. Consider S_0 and S as the supplier and the retailer inventory positions, respectively. We define (as in Axsäter’s papers for one retailer and where $S_0 \geq 0$) the following notations:

$g^{S_0}(t)$ = Density function of the Erlang (λ, S_0) distribution of the time that the warehouse orders a unit to it is demanded by the retailer, for the one-for-one corresponding system with Poisson demand (see Axsäter [16] page 65),

$G^{S_0}(t)$ = Cumulative distribution function of $g^{S_0}(t)$. thus:

$$g^{S_0}(t) = \frac{\lambda^{S_0} t^{S_0-1}}{(S_0-1)!} e^{-\lambda t}, \quad (\text{A1})$$

and:

$$G^{S_0}(t) = \sum_{k=S_0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \quad (\text{A2})$$

The average warehouse holding cost per unit is:

$$\begin{aligned} \gamma(S_0) &= \frac{h_0 S_0}{\lambda} (1 - G^{S_0+1}(L_0)) \\ &\quad - h_0 L_0 (1 - G^{S_0}(L_0)), \quad S_0 > 0, \end{aligned} \quad (\text{A3})$$

and for $S_0 = 0$,

$$\gamma(0) = 0. \quad (\text{A4})$$

Given the value of random delay at the warehouse is equal to $t(S_0 \geq 0 \rightarrow 0 \leq t \leq L_0)$, the conditional expected cost per unit at the retailer is:

$$\begin{aligned} \pi^S(t) &= e^{-\lambda(L+t)} \frac{h + \beta}{\lambda} \sum_{k=0}^{S-1} \frac{(s-k)}{k!} (L+t)^k \lambda^k \\ &\quad + \beta \left(L + t - \frac{S}{\lambda} \right), \quad S > 0, \end{aligned} \quad (\text{A5})$$

($0! = 1$ by definition) and:

$$\pi^S(t) = \beta \left(L + t - \frac{S}{\lambda} \right), \quad S \leq 0. \quad (\text{A6})$$

The expected retailer inventory carrying and shortage costs, which was incurred to fill a unit of demand at the retailer, is:

$$\begin{aligned} \Pi^S(S_0) &= \int_0^{L_0} g_0^{S_0}(L_0 - t) \pi^S(t) dt \\ &\quad + (1 - G^{S_0}(L_0)) \pi^S(0), \quad S_0 > 0, \end{aligned} \quad (\text{A7})$$

and:

$$\Pi^S(0) = \pi^S(L_0). \quad (\text{A8})$$

Furthermore, for $S > 0$ and for large values of S_0 we have:

$$\Pi^S(S_0) \approx \pi^S(0). \quad (\text{A9})$$

The procedure starts by determining \bar{S}_0 such that:

$$G^{\bar{S}_0}(L_0) < \varepsilon, \quad (\text{A10})$$

where ε is a small positive number.

The recursive computational procedure is:

$$\begin{aligned} \Pi^S(S_0 - 1) = & \Pi^{S-1}(S_0) + (1 - G^{S_0}(L_0)) \\ & \times (\pi^S(0) - \pi^{S-1}(0)), \quad S_0 > 0, \quad (\text{A11}) \end{aligned}$$

and for $S \leq 0$ it is possible to show that:

$$\begin{aligned} \Pi^S(S_0) = & G^{S_0}(L_0)\beta L_0 - G^{S_0+1}(L_0)\beta \frac{S_0}{\lambda} \\ & + \beta \left(L - \frac{S}{\lambda} \right), \quad S_0 > 0. \quad (\text{A12}) \end{aligned}$$

The sum of the expected total holding and shortage costs per time unit in an inventory system with a one-for-one ordering policy is:

$$c(S_0, S) = \lambda(\Pi^S(S_0) + \gamma(S_0)). \quad (\text{A13})$$