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Research Article

A robust uniformly convergent scheme for two parameters singularly perturbed parabolic problems with time delay

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Abstract

A singularly perturbed time delay parabolic problem with two small parameters is considered. The paper develops a finite difference scheme that is exponentially fitted on a uniform mesh in the spatial direction and uses the implicit-Euler method to discretize the time derivative in the temporal direction in order to obtain a better numerical approximation to the solutions of this class of problems. We establish the parameter-uniform error estimate and discuss the stability of the suggested approach. In order to demonstrate the improvement in terms of accuracy, numerical results are also shown to validate the theoretical conclusions and are contrasted with the current hybrid scheme.

AMS subject classifications (2020): Primary 65M06; Secondary 65M12, 65L11.

Keywords: Singular perturbation, Two parameters parabolic convection-diffusion problem, Time delay, Fitted operator scheme, the Error estimate

1 Introduction

We deal with the following class of singularly perturbed parabolic initial-boundary-value problems (IBVPs) on the domain $D = \Omega_x \times (0, T]$, $\Omega_x = (0, 1)$:

$$\begin{cases} \$_{\varepsilon,\mu} u(x,t) \equiv u_t - \varepsilon u_{xx} - \mu a(x,t)u_x + b(x,t)u = w(x,t), \\ u(x,t) = \phi_b(x,t), \qquad (x,t) \in \Gamma_b = [0,1] \times [-\tau, 0], \\ u(0,t) = \phi_l(t), \qquad \Gamma_l = \{(0,t) : 0 \le t \le T\}, \\ u(1,t) = \phi_r(t), \qquad \Gamma_r = \{(1,t) : 0 \le t \le T\}, \end{cases} \tag{1}$$

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where $w(x,t) = -c(x,t)u(x,t-\tau)+f(x,t), (x,t) \in D$. Here in this paper, $\Gamma = \Gamma_b \cup \Gamma_l \cup \Gamma_r$ with the parameters ε and μ such that $0 < \varepsilon \leq 1, 0 \leq \mu \leq 1$, and $\tau > 0$ represents the delay parameter and the functions $a(x,t), b(x,t), c(x,t), f(x,t)$, $\phi_b(x,t), \phi_l(t)$, and $\phi_r(t)$ are sufficiently smooth, bounded functions independent of ε and μ with

$$a(x,t) \geq \alpha > 0, \quad b(x,t) \geq \beta > 0, \quad c(x,t) \geq \vartheta > 0, \qquad (x,t) \in \overline{D} = [0,T] \times [0,1]\,.$$

The type of singularly perturbed two-parameter problems changes depending on the values of the perturbation parameters ε and μ; for $\mu = 0$, the problem is a reaction-diffusion problem, whereas, for $\mu = 1$, it is a convection-diffusion problem. It is well known that due to the presence of layers, classical numerical methods using a uniform mesh cannot properly approximate the exact solution when the parameter decreases unless a large number of mesh-intervals are utilized. However, even for lesser values of the perturbation parameters, one can overcome this difficulty by employing the fitted operator technique, which works without the prior location of the boundary layer. Time delay parabolic differential equations have recently attracted increasing amounts of attention due to their widespread use in many diverse application fields, including material science, biosciences, medicine, control theory, economics, and so on; see [20, 1, 10, 21, 23]. Many researchers have discussed the numerical results of the solutions of one-parameter singularly perturbed parabolic differential equations with time delay. For instance, one can refer to the articles by Das and Natesan [3], Gowrisankar and Natesan [6], Kumar [7], Woldaregay et al. [22], and Negero and Duressa [13, 14, 15, 16, 17].

In recent years, the development of a fitted numerical scheme for solving singularly perturbed time-delay parabolic problems having two parameters has received significant attention from a few authors. One such efficient fitted numerical scheme is an upwind difference scheme, which is proposed for solving singularly perturbed time-delay parabolic problems having two parameters in [5] by Govindarao, Mohapatra, and Sahu. They constructed a method on the Shishkin type meshes (standard Shishkin mesh, Bakhvalov-Shishkin mesh) and proved that the method is first-order accurate. Negero [12] considered the same problem in [5] and produced a second-order convergent scheme using an exponentially fitted cubic spline scheme. Prior to Negero's [12] strategy, there were no developed numerical techniques for addressing two-parameter singularly perturbed time-delayed parabolic problems based on fitted operators. Kumar et al. [8] devised and analyzed a hybrid monotone finite difference scheme for singularly perturbed IBVPs of the form (1). In [8], a first-order uniformly convergent method is given for problem (1) using a hybrid monotone finite difference scheme on a rectangular mesh, which is a combination of a uniform mesh in time and a layer-adapted Shishkin mesh in space. There were no established numerical methods for dealing with two-parameter singularly perturbed time-delay parabolic problems based on fitted operators prior to Negero's [12] strategy. Thus, the main aim of the present study is to provide robust parameter uniform convergent numerical methods based on exponentially fitted for the solution of problem (1).

Organization of the paper: In Section 2, the properties of the continuous solution are given. In Section 3, we describe the construction of an exponentially fitted finite difference discretization of problem (1). The stability and uniform convergence analysis of the suggested technique are given in Section 4. Some numerical results that validate our theory are reported in Section 5. Lastly, in Section 6, we present the conclusion of the paper.

Notations: The norm $\|\cdot\|$ is used to denote the maximum norm over the domain $\bar{D}$, defined as $\|g\| = \max_{\bar{D}} |g(x,t)|$ for a function g defined on some domain $\bar{D}$. In addition, C and its subscripts stand for positive constants independent of the perturbation parameters ε, μ, and mesh sizes.

2 Properties of the continuous solution

The required compatibility conditions at the corner points are

$$\begin{cases} \phi_b(0,0) = \phi_l(0), \\ \phi_b(1,0) = \phi_r(0), \end{cases} \tag{2}$$

$$\begin{cases} \dfrac{\partial \phi_l(0)}{\partial t} - \varepsilon \dfrac{\partial^2 \phi_b(0,0)}{\partial x^2} - \mu a(0,0) \dfrac{\partial \phi_b(0,0)}{\partial x} + b(0,0)\phi_b(0,0) \\ = -c(0,0)\phi_b(0,-\tau) + f(0,0), \\ \dfrac{\partial \phi_r(0)}{\partial t} - \varepsilon \dfrac{\partial^2 \phi_b(1,0)}{\partial x^2} - \mu a(1,0) \dfrac{\partial \phi_b(1,0)}{\partial x} + b(1,0)\phi_b(1,0) \\ = -c(1,0)\phi_b(0,-\tau) + f(1,0), \end{cases} \tag{3}$$

so that the data matches at the two corners $(0,0)$ and $(1,0)$. Let a, b, c, and f be continuous on a domain D. Then (1) has a unique solution $u \in C^2(D)$ [9].

Lemma 1 (Continuous maximum principle)**.** Let $\Phi(x,t) \in C^2(D) \cap C^0(\bar{D})$ and $\Phi(x,t) \geq 0$ for all $(x,t) \in \Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$. Then $\$_{\varepsilon,\mu}\pi(x,t) \geq 0$ in D gives $\Phi(x,t) \geq 0$, for all $(x,t) \in \bar{D}$.

Proof. Assume $(\theta^*, \zeta^*) \in D$ such that $\Phi(\theta^*, \zeta^*) = \min_{(x,t)\in\bar{D}} \Phi(x,t)$ and $\Phi(\theta^*, \zeta^*) < 0$. Then, it is easy to verify that $\$_{\varepsilon,\mu}\Phi(\theta^*, \zeta^*) < 0$, which is a contradiction. Thus, we have $\Phi(x,t) \geq 0$ for all $(x,t) \in \bar{D}$. □

Lemma 2. The solution $u(x,t)$ of the continuous problem (1) is bounded as

$$|u(x,t) - \phi_b(x,0)| \leq Ct.$$

Proof. Refer to [8]. □

Lemma 3. The bound on the solution $u(x,t)$ of the continuous problem (1) is given by

$$|u(x,t)| \leq C, \qquad (x,t) \in \bar{D}.$$

Proof. Refer to [8]. □

Lemma 4 (Uniform stability estimate)**.** Let $u(x,t)$ be the solution of the continuous problem in (1). Then we have the bound

$$\|u(x,t)\| \leq \beta^{-1}\|w\| + \max\left\{|\phi_b| + \max\left(|\phi_l|, |\phi_r|\right)\right\}.$$

Proof. An application of Lemma 1 to the comparison function

$$\chi^{\pm}(x,t) = \beta^{-1}\|g\| + \max\left(|\phi_b|, (|\phi_l| + |\phi_r|)\right) \pm u(x,t), \qquad (x,t) \in \bar{D},$$

yields the required estimate. □

Lemma 5. Let $u(x,t)$ be the solution of problem (1), satisfying $0 \leq i+2j \leq 4$. Then $u(x,t)$ satisfies the following bound:

$$\left\|\frac{\partial^{i+j}u}{\partial x^i \partial t^j}\right\| \leq C \begin{cases} \frac{1}{(\sqrt{\varepsilon})^i} & \text{when} \quad \alpha\mu^2 \leq \varepsilon\eta, \\ \left(\frac{\mu}{\varepsilon}\right)^i \left(\frac{\mu^2}{\varepsilon}\right)^j & \text{when} \quad \alpha\mu^2 \geq \varepsilon\eta, \end{cases}$$

where $\eta \approx \min_{(x,t)\in\bar{D}} \frac{b(x,t)}{a(x,t)}$.

Proof. Refer to [8]. □

3 Numerical scheme formulation

3.1 Temporal discretization

The time interval $[0,T]$ is partitioned into a uniform step size as follows:

$$\Omega_t^M = \{t_m = m\Delta t, m = 0,1,\ldots,M, \Delta t = T/M\}, \quad T = ks, \quad s = m_s\Delta t,$$

where k is a positive constant, m_s is a positive integer, Δt is the time step size, and M is the number of mesh intervals.

Hence, the problem (1) is discretized by using the implicit Euler method as follows:

$$\begin{cases} \dfrac{U^{m+1}(x)-U^{m}(x)}{\Delta t}-\varepsilon\left(U_{xx}\right)^{m+1}(x)-\mu a^{m+1}(x) \\ \left(U_{x}\right)^{m+1}(x)+b^{m+1}(x)\,U^{m+1}(x)=w^{m+1}(x), \\ U^{m}(0)=\phi_{l}\left(t_{m}\right), \quad 0\leq m\leq M, x\in\Omega_{x}, \\ U^{m}(1)=\phi_{r}\left(t_{m}\right), \quad 0\leq m\leq M, x\in\Omega_{x}, \\ U^{m+1}(x)=\phi_{b}\left(x,t_{m+1}\right), \quad -(s+1)\leq m\leq -1, \quad x\in\Omega_{x}, \end{cases} \tag{4}$$

where $w^{m+1}(x)=-c^{m+1}(x)U^{m+1-s}(x)+f^{m+1}(x)$, $0\leq m\leq M, x\in\Omega_x$ and $U^{m+1}(x)$ is the semidiscrete approximation to the exact solution $u(x,t_{m+1})$ of (1) at the $(m+1)th$ time level. Then, let us rewrite (4) in the following operator form:

$$\begin{cases} \$^{M}_{\varepsilon,\mu}U^{m+1}(x)=H\left(x,t_{m+1}\right), \\ U^{m+1}(0)=\phi_{l}\left(t_{m+1}\right), \qquad 0\leq m\leq M, \\ U^{m+1}(1)=\phi_{r}\left(t_{m+1}\right), \qquad 0\leq m\leq M, \quad x\in\Omega_{x}, \\ U^{m+1}(x)=\phi_{b}\left(x,t_{m+1}\right), \qquad -(s+1)\leq m\leq -1, \quad x\in\Omega_{x}, \end{cases} \tag{5}$$

where

$$\$^{M}_{\varepsilon,\mu}U^{m+1}(x)=-\varepsilon\left(U_{xx}\right)^{m+1}(x)-\mu a^{m+1}(x)\left(U_{x}\right)^{m+1}(x)+q^{m+1}(x)\,U^{m+1}(x)$$

and

$$H\left(x,t_{m+1}\right)=\frac{1}{\Delta t}U^{m}(x)-c^{m+1}(x)U^{m-s+1}(x)+f^{m+1}(x), \qquad 1\leq m\leq M, \quad x\in\Omega_{x},$$

for $q^{m+1}(x)=\dfrac{1}{\Delta t}+b^{m+1}(x)$.

Lemma 6 (Semidiscrete maximum principle)**.** Let $\varphi^{m+1}(x)\in C^{2}(D)\cap C^{0}\left(\bar{D}\right)$. If $\varphi^{m+1}(0)\geq 0$, $\varphi^{m+1}(1)\geq 0$, and $\$^{M}_{\varepsilon,\mu}\varphi^{m+1}(x)\geq 0$ for all $x\in D$, then $\varphi^{m+1}(x)\geq 0$ for all $x\in\bar{D}$.

Proof. One can prove this lemma by the same procedure as the proof of Lemma 1. ☐

Lemma 7 (Local error estimate)**.** Suppose $\dfrac{\partial^{i}u(x,t)}{\partial t^{i}}\leq C, (x,t)\in\bar{D}\times(0,T]$, $0\leq i\leq 2$. The local truncation error defined as $e_{m+1}=u\left(x,t_{m}\right)-U^{m}(x)$, associated to scheme (5) satisfies

$$\|e_{m+1}\|\leq C\left(\Delta t\right)^{2}, \qquad m=1,2,\ldots,M.$$

Proof. See [2]. ☐

Lemma 8 (Global error estimate.)**.** The global error E_{m+1} is estimated as

$$\|E_{m+1}\| \leq C\,(\Delta t)\,.$$

Proof. See [3]. □

At the $(n+1)th$ time level, the characteristics equation of the homogeneous part of the differential equation (5) can be

$$\varepsilon\lambda^2\,(x) + \mu a^{m+1}\,(x)\,\lambda\,(x) - \left(b^{m+1}\,(x) + \frac{1}{\Delta t}\right) = 0. \tag{6}$$

Then, the roots of (5) are

$$\lambda_1\,(x) = \frac{-\mu a^{m+1}(x)}{2\varepsilon} + \sqrt{\left(\frac{-\mu a^{m+1}(x)}{2\varepsilon}\right)^2 + \frac{\varrho^*}{\varepsilon}} > 0,$$
$$\lambda_2\,(x) = \frac{-\mu a^{m+1}(x)}{2\varepsilon} - \sqrt{\left(\frac{-\mu a^{m+1}(x)}{2\varepsilon}\right)^2 + \frac{\varrho^*}{\varepsilon}} < 0,$$

where $\varrho^* = b^{m+1}\,(x) + \dfrac{1}{\Delta t}$. From these roots, it is possible to see the boundary layer behavior of the solution in the neighborhood of $x = 0$ and $x = 1$. Let $\varrho_0 = -\max_{x\in[0,1]} \lambda_1\,(x)$ and $\varrho_1 = \min_{x\in[0,1]} \lambda_2\,(x)$. Then we have two cases
i) When $\frac{\mu^2}{\varepsilon} \to 0$, as $\varepsilon \to 0$, $\varrho_0 \approx \varrho_1 = \sqrt{\frac{\varrho_1^*}{\varepsilon}}$, where $0 < \varrho_1^* < \varrho^*$.
ii) When $\frac{\varepsilon}{\mu^2} \to 0$, as $\mu \to 0$, $= \frac{\mu}{\varepsilon}\varrho_2^*$ and $\varrho_1 = 0$, where $0 < \varrho_2^* < \mu a^{m+1(x)}$.
Next, we give the semidiscrete bound of the solution $U^{m+1}(x)$ of the problems in (6).

Lemma 9. [8]For a fixed number $0 < p < 1$ and for a certain order k, the solution $U^m(x)$ of (5) satisfies the following derivative bound

$$\left|\frac{d^i U^m(x)}{dx^i}\right| \leq C\left(1 + \varrho_0^{-i} e^{-p\varrho_0 x} + \varrho_1^{-i} e^{-p\varrho_1 (1-x)}\right), \qquad \text{for} \quad 0 \leq i \leq k.$$

3.2 Fully discrete problem

In this section, we fully discretize the problem under consideration via an exponentially fitted finite difference scheme for space derivative discretization. On the space domain $[0,1]$, we introduce the equidistant meshes with uniform mesh length h such that

$$\Omega_x^N = \{x_n = nh,\ n = 1, 2, \ldots, N, x_0 = 0, x_N = 1, h = 1/N\}\,,$$

where h is the step size, and N is the number of mesh points in the space direction. Using the theory applied in the asymptotic method developed in [18], we develop an exponentially fitted numerical scheme to solve the singularly perturbed BVPs in (6). In the considered case, the boundary layer is on the left side of the domain, so for the singularly perturbed problem of (6), the zero-order approximation asymptotic solution is given as

$$U^{m+1}(x) = U_0^{m+1}(x) + \left(\phi_l\left(t_{m+1}\right) - U_0^{m+1}(0)\right) \exp\left\{-\int_0^x \left(\frac{\mu a^{m+1}(x)}{\varepsilon}\right) dx\right\} + O(\varepsilon), \tag{7}$$

where $U_0^{m+1}(x)$ is the solution of the reduced problem in (6) obtained by setting $\varepsilon = 0$ written as

$$\begin{cases} \mu a^{m+1}(x) \dfrac{d}{dx} U_0^{m+1}(x) - q^{m+1}(x) U_0^{m+1}(x) = G^{m+1}(x), \\ U_0^{m+1}(0) = \phi_l\left(t_{m+1}\right). \end{cases} \tag{8}$$

Taking Taylor's series expansion for $a(x, t_m)$ about $x = 0$ and taking their first terms, (7) gives

$$U^{m+1}(x) = U_0^{m+1}(x) + \left(\phi_l\left(t_{m+1}\right) - U_0^{m+1}(0)\right) \exp\left\{-\left(\frac{\mu a^{m+1}(0)}{\varepsilon}\right) x\right\} + O(\varepsilon). \tag{9}$$

At the mesh $x_n = nh$, (9) becomes

$$U^{m+1}(nh) = U_0^{m+1}(nh) + \left(\phi_l\left(t_{m+1}\right) - U_0^{m+1}(0)\right) \exp\left\{-\left(\frac{\mu a^{m+1}(x)}{\varepsilon}\right)(nh)\right\} + O(\varepsilon).$$

Therefore,

$$\lim_{h\to 0} U^{m+1}(nh) = U_0^{m+1}(0) + \left(\phi_l\left(t_{m+1}\right) - U_0^{m+1}(0)\right) \exp\left\{-\mu a^{m+1}(0)\, n\rho\right\}, \tag{10}$$

where $\rho = \frac{\mu h}{\varepsilon}$.

Now, we consider the derivative approximation of the problem in (1) and (2) as

$$D^- U_n = \frac{U_n - U_{n-1}}{h},\ D^+ U_n = \frac{U_{n+1} - U_n}{h},\ D^0 U_n = \frac{U_{n+1} - U_{n-1}}{2h}, \text{ and}$$
$$D^+ D^- U_n = \frac{U_{n+1} - 2U_n + U_{n-1}}{h^2},$$

and

$$\varepsilon\sigma\left(\rho,\varepsilon,\mu\right)D^{+}D^{-}U^{m+1}\left(x_{n}\right)+\mu a^{m+1}\left(x_{n}\right)D^{0}U^{m+1}\left(x_{n}\right) \\ -q^{m+1}\left(x_{n}\right)U^{m+1}\left(x_{n}\right)=G\left(x_{n},t_{m+1}\right), \tag{11}$$

where $\sigma\left(\rho,\varepsilon,\mu\right)$ is a fitting factor.
Multiplying (11) by h and evaluating the limit as $h\to 0$ give

$$\lim_{h\to 0}\left[\frac{\sigma\left(\rho,\varepsilon,\mu\right)}{\rho}\left(U_{n+1}^{m+1}-2U_{n}^{m+1}+U_{n-1}^{m+1}\right)\right] \\ +\frac{1}{2}a^{m+1}\left(nh\right)\left(U_{n+1}^{m+1}-U_{n-1}^{m+1}\right)=0. \tag{12}$$

Substituting (10) into (12) and taking $a(x,t)=a$ constant with some manipulation give the fitting factor as

$$\sigma\left(\rho,\varepsilon,\mu\right)=a^{m+1}\left(0\right)\tfrac{\rho}{2}\coth\left(\tfrac{\rho a^{m+1}(0)}{2}\right).$$

For the variable fitting factor, we define as

$$\sigma_{n}\left(\rho,\varepsilon,\mu\right)=a^{m+1}\left(x_{n}\right)\frac{\rho}{2}\coth\left(\frac{\rho a^{m+1}\left(x_{n}\right)}{2}\right). \tag{13}$$

Hence, using (12) ,the resulting finite difference scheme can be given as

$$\begin{aligned}\$_{\varepsilon,\mu}^{N,M}U_{m}^{n+1}\equiv&\left(\frac{\varepsilon\sigma_{n}\left(\rho,\varepsilon,\mu\right)}{h^{2}}-\frac{1}{2}\mu a_{n}^{m+1}\right)U_{n-1}^{m+1}\\&+\left(\frac{-2\varepsilon\sigma_{n}\left(\rho,\varepsilon,\mu\right)}{h^{2}}-q_{n}^{m+1}\right)U_{n}^{m+1}\\&+\left(\frac{\varepsilon\sigma_{n}\left(\rho,\varepsilon,\mu\right)}{h^{2}}+\frac{1}{2}\mu a_{n}^{m+1}\right)U_{n+1}^{m+1}\\=&H_{n}^{m+1}\end{aligned} \tag{14}$$

subject to the following conditions:

$$\begin{cases}U_{0}^{m+1}=\phi_{l}\left(t_{m+1}\right), & 0\le m\le M,\\ U_{N}^{m+1}=\phi_{r}\left(t_{m+1}\right), & 0\le m\le M,\\ U\left(x_{n},t_{m+1}\right)=\phi_{b}\left(x_{n},t_{m+1}\right),x_{n}\in\bar{\Omega}^{N}, & -\left(\wp+1\right)\le m\le -1,\end{cases} \tag{15}$$

where

$$\begin{aligned}H_{n}^{m+1}&=H\left(x_{n},t_{m+1}\right)\\&=-\frac{1}{\Delta t}U^{m}\left(x_{n}\right)+c^{m+1}(x_{n})U^{m-\wp+1}\left(x_{n}\right)-f^{m+1}(x_{n}).\end{aligned}$$

The schemes in (14) and (15) can be rewritten as

$$\$_{\varepsilon,\mu}^{N,M}U_{n}^{m+1}\equiv E_{n}^{m+1}U_{n-1}^{m+1}-F_{n}^{m+1}U_{n}^{m+1}+G_{n}^{m+1}U_{n-1}^{m+1}=H_{n}^{m+1}, \tag{16}$$

where

$$E_n^{m+1} = \frac{\varepsilon\sigma_n(\rho,\varepsilon,\mu)}{h^2} - \frac{1}{2}\mu a_n^{m+1},$$
$$F_n^{m+1} = \frac{2\varepsilon\sigma_n(\rho,\varepsilon,\mu)}{h^2} + q_n^{m+1},$$
$$G_n^{m+1} = \frac{\varepsilon\sigma_n(\rho,\varepsilon,\mu)}{h^2} + \frac{1}{2}\mu a_n^{m+1},$$
$$H_n^{m+1} = -\frac{1}{\Delta t}U^m(x_n) + c^{m+1}(x_n)U^{m-\wp+1}(x_n) - f^{m+1}(x_n).$$

From the entries $E_n^{m+1}, F_n^{m+1}, G_n^{m+1}$ of tridiagonal system of (16), it is evident that $E_n^{m+1} < 0, G_n^{m+1} < 0$ and $E_n^{m+1} + F_n^{m+1} + G_n^{m+1} > 0$. Thus the system is an M-matrix, and therefore its inverse exists, and it is positive. Hence, the tridiagonal system in (16) can be easily solved by any existing methods.

4 Stability and uniform convergence analysis

Lemma 10 (Discrete maximum principle)**.** Assume that ψ_n^{m+1} is any mesh function that satisfies $\psi_0^{m+1} \geq 0$, $\psi_N^{m+1} \geq 0$, and that $\$_{\varepsilon,\mu}^{N,M}$ is the discrete operator of (16). Then $\$_{\varepsilon,\mu}^{N,M}\psi_n^{m+1} \geq 0$, for $1 \leq n \leq N-1$, implies that $\psi_n^{m+1} \geq 0$, for $0 \leq n \leq N$.

Proof. Refer to [12]. □

Lemma 11 (Uniform stability estimate for discrete problem)**.** Let U_n^{m+1} be any mesh function such that $U_0^{m+1} = 0, U_N^{m+1} = 0$ on $0 \leq n \leq N$. Then

$$\left|U_n^{m+1}\right| \leq \frac{\max\left|\$_{\varepsilon,\mu}^{N,M}U_n^{m+1}\right|}{q^*} + C\max\left\{\left|\phi_l(t_{m+1})\right|, \left|\phi_r(t_{m+1})\right|\right\},$$

where $q_n^{m+1} = \dfrac{1}{\Delta t} + b(x_n, t_{m+1}) \geq q^* > 0$.

Proof. Refer to [12]. □

Theorem 1. Let $U(x_n, t_{m+1})$ be the continuous solution of (1) and (2) and let U_n^{m+1} be the approximate solution of (16). Then, for sufficiently large N, the following error bound holds:

$$\left|\$_{\varepsilon,\mu}^{N,M}\left(U(x_n,t_{m+1}) - U_n^{m+1}\right)\right| \leq CN^{-2}.$$

Proof. Consider the error bound in the spatial direction as

$$\begin{aligned}
&\left|\$^{N,M}_{\varepsilon,\mu}\left(U\left(x_n,t_{m+1}\right)-U_n^{m+1}\right)\right| \\
&\quad=\left|\$^{N,M}_{\varepsilon,\mu}U\left(x_n,t_{m+1}\right)-\$^{N,M}_{\varepsilon,\mu}U_n^{m+1}\right| \\
&\quad=\Bigg|\varepsilon\left(U_{xx}\right)^{m+1}\left(x_n\right)+\mu a^{m+1}\left(x_n\right)\left(U_x\right)^{m+1}\left(x_n\right) \\
&\qquad-\left\{\varepsilon\sigma\left(\rho,\varepsilon,\mu\right)D^+D^-U_n^{m+1}+\mu a_n^{m+1}D^0U_n^{m+1}\right\}\Bigg| \\
&\quad\leq\left|\varepsilon\sigma\left(\rho,\varepsilon,\mu\right)\left(\frac{d^2}{dx^2}-D^+D^-\right)U_n^{m+1}+\mu a_n^{m+1}\left(\frac{d}{dx}-D^0\right)U_n^{m+1}\right| \\
&\quad\leq\left|\varepsilon\left[a^{m+1}\left(x_n\right)\frac{\rho\mu}{2}\coth\left(\frac{\rho\mu a^{m+1}\left(x_n\right)}{2}\right)-1\right]D^+D^-U_n^{m+1}\right| \\
&\qquad+\left|\varepsilon\left(\frac{d^2}{dx^2}-D^+D^-\right)U_n^{m+1}\right|+\left|\mu a_n^{m+1}\left(\frac{d}{dx}-D^0\right)U_n^{m+1}\right|.
\end{aligned} \tag{17}$$

Now, (17) becomes

$$\begin{aligned}
&\left|\$^{N,M}_{\varepsilon,\mu}\left(U\left(x_n,t_{m+1}\right)-U_n^{m+1}\right)\right| \\
&\qquad\leq C\mu h^2\frac{d^2U_n^{m+1}}{dx^2}+C\varepsilon h^2\frac{d^4U_n^{m+1}}{dx^4}+C\mu h^2\frac{d^3U_n^{m+1}}{dx^3}.
\end{aligned}$$

Using Lemma 9, we have

$$\begin{aligned}
&\left|\$^{N,M}_{\varepsilon,\mu}\left(U\left(x_n,t_{m+1}\right)-U_n^{m+1}\right)\right| \\
&\qquad\leq C\mu h^2\left(1+\omega_1^{-2}e^{-\nu\omega_1x}+\omega_2^{-2}e^{-\nu\omega_2(1-x)}\right) \\
&\qquad\quad+Ch^2\Bigg[\varepsilon\left(1+\omega_1^{-4}e^{-\nu\omega_1x}+\omega_2^{-4}e^{-\nu\omega_2(1-x)}\right) \\
&\qquad\quad+\mu\left(1+\omega_1^{-3}e^{-\nu\omega_1x}+\omega_2^{-3}e^{-\nu\omega_2(1-x)}\right)\Bigg].
\end{aligned}$$

As $\varepsilon\to 0$ both $\omega_1^{-i}e^{-\nu\omega_1x_m}$ and $\omega_2^{-i}e^{-\nu\omega_2(1-x_m)}$ approach zero for $0\leq i\leq 4$. Thus, we obtain the following error bound:

$$\left|\$^{N,M}_{\varepsilon}\left(U\left(x_n,t_{m+1}\right)-U_n^{m+1}\right)\right|\leq CN^{-2},$$

since $h=N^{-1}$. □

Under the hypothesis of Lemmas 11 and 10, the following error estimate holds:

$$\max_{0\leq n<N}\left|U\left(x_n,t_{m+1}\right)-U_n^{m+1}\right|\leq Ch=CN^{-2}. \tag{18}$$

Theorem 2. Let $u(x,t)$ be the exact solution of (1) and (2) and let U_n^{m+1} be the numerical solution of (16). For the discrete scheme, there exist a constant C independent of ε, h and Δt such that

$$\max_{0\leq n\leq N, 0\leq m\leq M} \left|u(x_n, t_{m+1}) - U_n^{m+1}\right| \leq C\left(N^{-2} + (\Delta t)\right).$$

for sufficiently large N.

Proof. The result follows from the error estimate given in Lemma 8 and Theorem 1. □

5 Numerical results

In this section, we illustrate the proposed scheme using two numerical examples of the form given in (1). We investigate the theoretical results in this paper by performing experiments using the proposed scheme. The exact solution of these two examples is not known. Thus, we use the double mesh principle to evaluate maximum absolute errors $E_{\varepsilon,\mu}^{N,M}$ and the corresponding order of convergence $p_{\varepsilon,\mu}^{N,M}$ as

$$E_{\varepsilon,\mu}^{N,M} = \max_{0\leq n\leq N, 0\leq m\leq M} \left|U_n^{m+1} - U_{2n}^{2m+1}\right|, \qquad p_{\varepsilon,\mu}^{N,M} = \log_2\left(\frac{E_{\varepsilon,\mu}^{N,M}}{E_{\varepsilon,\mu}^{2N,2M}}\right).$$

From these values, we obtain the ε-uniform error $E^{N,M}$ and the corresponding ε-uniform order of convergence $p^{N,M}$ as

$$E^{N,M} = \max_{0\leq n\leq N, 0\leq m\leq M} E_{\varepsilon}^{N,M} \text{ and } p^{N,M} = \log_2\left(\frac{E^{N,M}}{E^{2N,2M}}\right),$$

where U_m^{n+1} is the numerical solutions obtained by using $N\times M$ mesh intervals in space and time direction with mesh size h and Δt, respectively.

Example 1. Consider the problem

$$\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} - \mu(1+x)\frac{\partial u}{\partial x} + u(x,t) = -u(x,t-\tau) + 16x^2(1-x)^2, \quad (x,t)\in(0,1)\times(0,2],$$

with

$$\begin{cases} u(0,t) = 0, \qquad u(1,t) = 0, \qquad t\in(0,2], \\ u(x,t) = 0, \qquad (x,t)\in[0,1]\times[-\tau,0]. \end{cases}$$

Example 2. Consider the problem

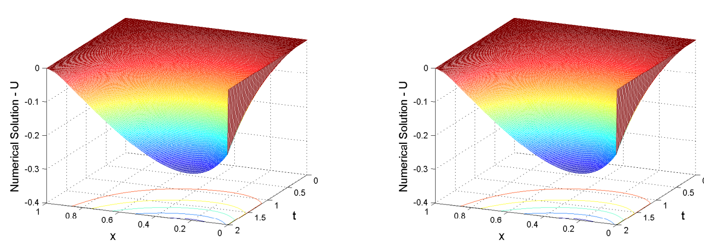
$$\frac{\partial u}{\partial t}-\varepsilon\frac{\partial^2 u}{\partial x^2}-\mu\left(1+x\left(1-x\right)+t^2\right)\frac{\partial u}{\partial x}+\left(1+5xt\right)u(x,t)$$
$$=-u(x,t-\tau)+x(1-x)\left(e^t-1\right), \qquad (x,t)\in(0,1)\times(0,2],$$

with

$$\begin{cases} u(0,t)=0, \qquad u(1,t)=0, \qquad t\in(0,2], \\ u(x,t)=0, \qquad (x,t)\in[0,1]\times[-\tau,0]. \end{cases}$$

Table 1: Maximum pointwise errors ($E^{N,M}_{\varepsilon,\mu}$) and rate of convergence ($p^{N,M}_{\varepsilon,\mu}$) for Example 1.

$\mu=10^{-4}$ $\varepsilon\downarrow$	N=32 M=16	N=64 M=32	N=128 M=64	N=256 M=128	N=512 M=256
10^{-0}	5.7516e-03	3.6382e-03	2.1286e-03	1.1627e-03	6.0947e-04
	0.66074	0.77332	0.87243	0.93185	-
10^{-2}	1.0422e-02	5.4491e-03	2.7875e-03	1.4101e-03	7.0919e-04
	0.93554	0.96705	0.98317	0.99155	-
10^{-4}	1.0658e-02	5.5312e-03	2.8189e-03	1.4231e-03	7.1502e-04
	0.94627	0.97246	0.98610	0.99298	-
10^{-6}	1.0663e-02	5.5328e-03	2.8193e-03	1.4232e-03	7.1504e-04
	0.94653	0.97267	0.98620	0.99304	-
10^{-8}	1.0664e-02	5.5339e-03	2.8202e-03	1.4237e-03	7.1533e-04
	0.94638	0.97250	0.98615	0.99296	-
10^{-10}	1.0664e-02	5.5339e-03	2.8202e-03	1.4237e-03	7.1533e-04
	0.94638	0.97250	0.98615	0.99296	-
10^{-12}	1.0664e-02	5.5339e-03	2.8202e-03	1.4237e-03	7.1533e-04
	0.94638	0.97250	0.98615	0.99296	-
$E^{N,M}$	**1.0664e-02**	**5.5339e-03**	**2.8202e-03**	**1.4237e-03**	**7.1533e-04**
$p^{N,M}$	**0.94638**	**0.97250**	**0.98615**	**0.99296**	-



(a) (b)

Figure 1: Surface plot of the numerical solution for Example 2 with $N=256, M=128$, **a** $\varepsilon=10^{-1}$,$\mu=10^{-12}$ **b** $\varepsilon=10^{-12}$,$\mu=10^{-1}$

Table 2: Maximum pointwise errors ($E^{N,M}_{\varepsilon,\mu}$) and rate of convergence ($p^{N,M}_{\varepsilon,\mu}$) for Example 1.

$\mu=10^{-12}$	Number of mesh intervals N=M				
$\varepsilon\downarrow$	32	64	128	256	512
10^{-0}	3.6218e-03	2.1253e-03	1.1619e-03	6.0925e-04	3.1225e-04
	0.57919	0.87118	0.93138	0.96433	-
10^{-2}	5.4110e-03	2.7780e-03	1.4077e-03	7.0860e-04	3.5549e-04
	0.96185	0.98071	0.99030	0.99516	-
10^{-4}	5.5307e-03	2.8187e-03	1.4231e-03	7.1501e-04	3.5838e-04
	0.97243	0.98599	0.99300	0.99647	-
10^{-6}	5.5315e-03	2.8189e-03	1.4231e-03	7.1502e-04	3.5839e-04
	0.97254	0.98610	0.99298	0.99645	-
10^{-8}	5.5315e-03	2.8189e-03	1.4231e-03	7.1502e-04	3.5838e-04
	0.97254	0.98610	0.99298	0.99645	-
10^{-10}	5.5315e-03	2.8189e-03	1.4231e-03	7.1502e-04	3.5838e-04
	0.97254	0.98610	0.99298	0.99645	-
10^{-12}	5.5315e-03	2.8189e-03	1.4231e-03	7.1502e-04	3.5838e-04
	0.97254	0.98610	0.99298	0.99645	-
$E^{N,M}$	**5.5315e-03**	**2.8189e-03**	**1.4231e-03**	**7.1502e-04**	**3.5839e-04**
$p^{N,M}$	**0.97254**	**0.98610**	**0.99298**	**0.99645**	-

Table 3: Maximum pointwise errors ($E^{N,M}_{\varepsilon,\mu}$) and rate of convergence ($p^{N,M}_{\varepsilon,\mu}$) for Example 2.

$\mu=10^{-4}$	N=32	N=64	N=128	N=256	N=512
$\varepsilon\downarrow$	M=16	M=32	M=64	M=128	M=256
10^{-0}	2.1475e-04	1.0912e-04	5.4962e-05	2.7578e-05	1.3813e-05
	0.97674	0.98941	0.99492	0.99749	-
10^{-2}	2.1465e-03	1.1561e-03	6.0053e-04	3.0592e-04	1.5440e-04
	0.89272	0.94496	0.97308	0.98648	-
10^{-4}	2.6764e-03	1.4488e-03	7.5345e-04	3.8424e-04	1.9401e-04
	0.88544	0.94327	0.97150	0.98588	-
10^{-6}	2.6771e-03	1.4491e-03	7.5407e-04	3.8484e-04	1.9445e-04
	0.88551	0.94239	0.97044	0.98486	-
10^{-8}	2.6771e-03	1.4490e-03	7.5351e-04	3.8423e-04	1.9401e-04
	0.88561	0.94336	0.97166	0. 98584	-
10^{-10}	2.6771e-03	1.4490e-03	7.5351e-04	3.8423e-04	1.9401e-04
	0.88561	0.94336	0.97166	0. 98584	-
10^{-12}	2.6771e-03	1.4490e-03	7.5351e-04	3.8423e-04	1.9401e-04
	0.88561	0.94336	0.97166	0. 98584	-
$E^{N,M}$	**2.6771e-03**	**1.4491e-03**	**7.5407e-04**	**3.8424e-04**	**1.9445e-04**
$p^{N,M}$	**0.88551**	**0.94239**	**0.97269**	**0.98261**	-

Table 4: Maximum pointwise errors ($E_{\varepsilon,\mu}^{N,M}$) and rate of convergence ($p_{\varepsilon,\mu}^{N,M}$) for Example 2.

$\mu = 10^{-12}$	Number of mesh intervals N=M				
$\varepsilon \downarrow$	32	64	128	256	512
10^{-0}	1.3372e-04	6.1251e-05	2.9166e-05	1.4212e-05	7.0123e-06
	1.1264	1.0704	1.0372	1.0191	-
10^{-2}	1.1701e-03	6.0326e-04	3.0645e-04	1.5447e-04	7.7560e-05
	0.95578	0.97713	0.98833	0.99394	-
10^{-4}	1.4466e-03	7.5262e-04	3.8382e-04	1.9386e-04	9.7408e-05
	0.94267	0.97149	0.98541	0.99290	-
10^{-6}	1.4522e-03	7.5525e-04	3.8509e-04	1.9444e-04	9.7702e-05
	0.94321	0.97176	0.98587	0.99287	-
10^{-8}	1.4522e-03	7.5527e-04	3.8510e-04	1.9445e-04	9.7705e-05
	0.94318	0.97176	0.98583	0.99289	-
10^{-10}	1.4523e-03	7.5527e-04	3.8510e-04	1.9445e-04	9.7705e-05
	0.94328	0.97176	0.98583	0.99289	-
10^{-12}	1.4523e-03	7.5527e-04	3.8510e-04	1.9445e-04	9.7705e-05
	0.94328	0.97176	0.98583	0.99289	-
$E^{N,M}$	**1.4523e-03**	**7.5527e-04**	**3.8510e-04**	**1.9445e-04**	**9.7705e-05**
$p^{N,M}$	**0.94328**	**0.97176**	**0.98583**	**0.99289**	-

Table 5: Comparison of uniform error ($E^{N,M}$) for Example 1.

$\mu = 10^{-3}$	N=32	N=64	N=128	N=256	N=512
$\varepsilon \downarrow$	M=8	M=16	M=32	M=64	M=128
Proposed method					
10^{-4}	1.9859e-02	1.0660e-02	5.5318e-03	2.8190e-03	1.4232e-03
10^{-6}	1.9905e-02	1.0684e-02	5.5439e-03	2.8245e-03	1.4251e-03
10^{-8}	1.9905e-02	1.0684e-02	5.5440e-03	2.8252e-03	1.4262e-03
10^{-10}	1.9905e-02	1.0684e-02	5.5440e-03	2.8252e-03	1.4262e-03
10^{-12}	1.9905e-02	1.0684e-02	5.5440e-03	2.8252e-03	1.4262e-03
Method in [8]					
10^{-4}	4.3705e-2	1.6704e-2	7.3802e-3	3.7406e-3	1.8967e-3
10^{-6}	4.3471e-2	1.6596e-2	7.3290e-3	3.7218e-3	1.8873e-3
10^{-8}	4.3429e-2	1.6573e-2	7.3303e-3	3.7211e-3	1.8870e-3
10^{-10}	4.4343e-2	1.6572e-2	7.3303e-3	3.7211e-3	1.8870e-3
10^{-12}	4.4343e2	1.6572e-2	7.3303e-3	3.7211e-3	1.8870e-3

Table 6: Comparison of uniform error ($E^{N,M}$) for Example 2.

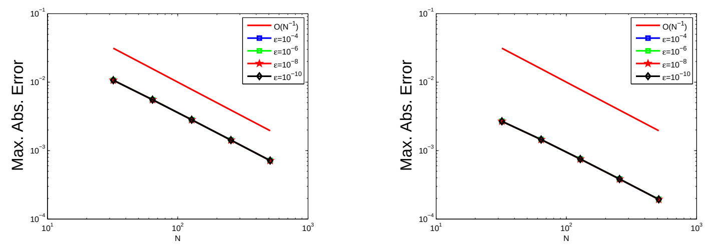
$\mu = 10^{-3}$ $\varepsilon \downarrow$	N=32 M=8	N=64 M=16	N=128 M=32	N=256 M=64	N=512 M=128
Proposed method					
10^{-4}	4.5627e-03	2.6876e-03	1.4564e-03	7.5785e-04	3.8653e-04
10^{-6}	4.5254e-03	2.6603e-03	1.4402e-03	7.4904e-04	3.8203e-04
10^{-8}	4.5254e-03	2.6603e-03	1.4402e-03	7.4904e-04	3.8195e-04
10^{-10}	4.5254e-03	2.6603e-03	1.4402e-03	7.4904e-04	3.8195e-04
10^{-12}	4.5254e-03	2.6603e-03	1.4402e-03	7.4904e-04	3.8195e-04
Method in [8]					
10^{-4}	1.1161e-2	5.1087e-3	2.4749e-3	1.2214e-3	6.0706e-4
10^{-6}	1.1008e-2	5.0450e-3	2.4437e-3	1.2073e-3	6.0036e-4
10^{-8}	1.0941e-2	5.0426e-3	2.4442e-3	1.2071e-3	6.0016e-4
10^{-10}	1.0940e-2	5.0428e-3	2.4442e-3	1.2071e-3	6.0016e-4
10^{-12}	1.0940e-2	5.0428e-3	2.4442e-3	1.2071e-3	6.0016e-4

Table 7: Comparison of uniform error ($E^{N,M}$) for Example 1.

$\mu = 10^{-9}$ $\varepsilon \downarrow$	N=32 M=8	N=64 M=16	N=128 M=32	N=256 M=64	N=512 M=128
Proposed method					
10^{-4}	1.9853e-02	1.0658e-02	5.5313e-03	2.8189e-03	1.4231e-03
10^{-6}	1.9856e-02	1.0659e-02	5.5315e-03	2.8189e-03	1.4231e-03
10^{-8}	1.9856e-02	1.0659e-02	5.5315e-03	2.8189e-03	1.4231e-03
10^{-10}	1.9856e-02	1.0659e-02	5.5315e-03	2.8189e-03	1.4231e-03
10^{-12}	1.9856e-02	1.0659e-02	5.5315e-03	2.8189e-03	1.4231e-03
Method in [8]					
10^{-4}	4.3708e-2	1.6705e-2	7.3807e-3	3.7407e-3	1.8967e-3
10^{-6}	4.3816e-2	1.6749e-2	7.4017e-3	3.7489e-3	1.9008e-3
10^{-8}	4.3817e-2	1.6750e-2	7.4019e-3	3.7490e-3	1.9008e-3
10^{-10}	4.3817e-2	1.6750e-2	7.4019e-3	3.7490e-3	1.9008e-3
10^{-12}	4.3817e-2	1.6750e-2	7.4019e-3	3.7490e-3	1.9008e-3

Table 8: Comparison of uniform error ($E^{N,M}$) for Example 2.

$\mu = 10^{-9}$ $\varepsilon \downarrow$	N=32 M=8	N=64 M=16	N=128 M=32	N=256 M=64	N=512 M=128
Proposed method					
10^{-4}	4.5499e-03	2.6744e-03	1.4477e-03	7.5307e-04	3.8398e-04
10^{-6}	4.5651e-03	2.6830e-03	1.4523e-03	7.5527e-04	3.8513e-04
10^{-8}	4.5652e-03	2.6831e-03	1.4523e-03	7.5529e-04	3.8514e-04
10^{-10}	4.5652e-03	2.6831e-03	1.4523e-03	7.5529e-04	3.8514e-04
10^{-12}	4.5652e-03	2.6831e-03	1.4523e-03	7.5529e-04	3.8514e-04
Method in [8]					
10^{-4}	1.1053e-2	5.0755e-3	2.4578e-3	1.2132e-3	6.0309e-4
10^{-6}	1.1046e-2	5.0765e-3	2.4625e-3	1.2161e-3	6.0456e-4
10^{-8}	1.1100e-2	5.0838e-3	2.4627e-3	1.2161e-3	6.0457e-4
10^{-10}	1.1093e-2	5.0782e-3	2.4639e-3	1.2162e-3	6.0457e-4
10^{-12}	1.1092e-2	5.0775e-3	2.4640e-3	1.2162e-3	6.0457e-4



(a) (b)

Figure 2: Log-Log plot of the maximum error on left (a) for Example 1 with $\mu = 10^{-4}$ and on right (b) for Example 2 with $\mu = 10^{-4}$.

We have demonstrated maximum pointwise errors ($E_{\varepsilon,\mu}^{N,M}$) and the rate of convergence ($p_{\varepsilon,\mu}^{N,M}$) for Example 1 using scheme (16) by fixing $\mu = 10^{-4}$ in Table 1 and $\mu = 10^{-12}$ in Table 2 with various values of ε. Similarly, Tables 3 and 4 have presented the result obtained for Example 2. The results given in Tables 1–4 clearly indicate that the proposed numerical method is accurate of order $O\left(N^{-2} + \Delta t\right)$, which approves the hypothetical result predicted in the theory. Numerical solutions obtained by the presented numerical scheme (16) for Example 2 have been shown in Figure 1 (a), (b), and it shows the effects of the two parameters ε and μ on the steepness of the layer of the solutions. From Figure 1 (a), we confirm the nonoccurrence of both left and right boundary layers near $x = 0$ and $x = 1$ for $\mu \to 0$ as ε becomes large. Similarly, from Figure 1 (b), we confirm the occurrence of left boundary layers near $x = 0$ for $\mu = 1$ as ε becomes small. The graphs between N and maximum pointwise errors of Examples 1 and 2 are plotted as the log-log scale, respectively, in Figure 2 (a) and (b). From these two graphs, one can observe that the numerical scheme converges ε-uniformly as the perturbation parameter goes very small. The comparison of our numerical results with that of [8] is presented in Tables 5–8. From these tables, we can confirm the improved accuracy of our proposed numerical method.

6 Conclusion

A singularly perturbed parabolic differential equation exhibiting boundary layers was considered. The considered problem contains two small perturbation parameters multiplied by the highest order derivative a term of the equation and a large delay parameter on the time variable. An exponentially fitted operator numerical scheme was proposed for solving the problem. First, the equation was approximated by equivalent singularly perturbed parabolic partial differential equations using the implicit Euler method in the time direction. Inducing an exponential fitting factor for a term with the perturbation parameter ε and determining its value, a fully discrete numerical scheme was developed using implicit Euler in temporal discretization and the central finite difference method for spatial discretization. The uniform stability and uniform convergence of the scheme were established. It was shown that the scheme is accurate and converges uniformly with the order of convergence $O\left(N^{-2}+(\Delta t)\right)$.

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References

[1] Asl, F.M., and Ulsoy, A.G. *Analysis of a system of linear delay differential equations*, J. Dyn. Sys., Meas., Control, 125 (2) (2003), 215–223.

[2] Clavero, C., Jorge, J. and Lisbona, F. *A uniformly convergent scheme on a nonuniform mesh for convection-diffusion parabolic problems.* J. Comput. Appl. Math. 154(2) (2003), 415–429.

[3] Das, A. and Natesan, S. *Uniformly convergent hybrid numerical scheme for singularly perturbed delay parabolic convection-diffusion problems on Shishkin mesh*, Appl. Math. Comput. 271 (2015), 168–186.

[4] Epstein, I.R. *Delay effects and differential delay equations in chemical kinetics*, Int. Rev. Phys. Chem. 11 (1) (1992), 135–160.

[5] Govindarao, L., Sahu, S.R. and Mohapatra, J. *Uniformly convergent numerical method for singularly perturbed time delay parabolic problem with two small parameters*, Iran. J. Sci. Technol. Trans. A Sci. 43(5) (2019), 2373–2383.

[6] Gowrisankar, S. and Natesan, S. *ε- uniformly convergent numerical scheme for singularly perturbed delay parabolic partial differential equations*, Int. J. Comput. Math. 94 (2017), 902–921.

[7] Kumar, D. *A parameter-uniform scheme for the parabolic singularly perturbed problem with a delay in time*, Numer. Methods Partial Differ. Equ. 37 (1) (2021), 626–642.

[8] Kumar, S. and Kumar, M. *A robust numerical method for a two-parameter singularly perturbed time delay parabolic problem*, Comput. Appl. Math. 39(3) (2020), 1–25.

[9] Ladyzhenskaia, O.A., Solonnikov, V.A. and Ural'tseva, N.N. *Linear and quasilinear equations of parabolic type.*(Russian) Translated from the Russian by S. Smith Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I. 1968.

[10] McCartin, B.J. *Discretization of the semiconductor device equations,*"New problems and new solutions for device and process modeling", Boole, (1985), 72–82.

[11] Miller, J., O'Riordan, E., Shishkin, G. and Shishkina, L. *Fitted mesh methods for problems with parabolic boundary layers*, Math. Proc. R. Ir. Acad. 98A (1998), no. 2, 173–190.

[12] Negero, N.T. *A uniformly convergent numerical scheme for two parameters singularly perturbed parabolic convection-diffusion problems with a large temporal lag*, Results Appl. Math. 16 (2022), Paper No. 100338, 15 pp.

[13] Negero, N.T. and Duressa, G.F. *A method of line with improved accuracy for singularly perturbed parabolic convection-diffusion problems with large temporal lag*, Results Appl. Math. 11 (2021), 100174, 13 pp.

[14] Negero, N.T. and Duressa, G.F. *An efficient numerical approach for singularly perturbed parabolic convection-diffusion problems with large time-lag*, J. Math. Model. 10(2) (2022), 173–110.

[15] Negero, N.T. and Duressa, G.F. *Uniform convergent solution of singularly perturbed parabolic differential equations with general temporal-lag*, Iran. J. Sci. Technol. Trans. A Sci. 46(2) (2022), 507–524.

[16] Negero, N.T. and Duressa, G.F. *An exponentially fitted spline method for singularly perturbed parabolic convection-diffusion problems with large time delay*, Tamkang J. Math. (2022).

[17] Negero, N.T. and Duressa, G.F. *Parameter-uniform robust scheme for singularly perturbed parabolic convection-diffusion problems with large time-lag*, Comput. Methods Differ. Equ. 10 (4) (2022), 954–968.

[18] O'Malley Jr, R.E. *Introduction to singular perturbations,* Applied Mathematics and Mechanics, Vol. 14. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1974.

[19] Roos, H.G. and Uzelac, Z. *The SDFEM for a convection-diffusion problem with two small parameters*, Dedicated to John J. H. Miller on the occasion of his 65th birthday. Comput. Methods Appl. Math. 3(3) (2003), 443–458.

[20] Tikhonov, A.N. and Samarskii, A.A. *Equations of mathematical physics*, Courier Corporation, 2013.

[21] Van Harten, A. and Schumacher, J. *On a class of partial functional differential equations arising in feed-back control theory*, Differential equations and applications (Proc. Third Scheveningen Conf., Scheveningen, (1977), pp. 161–179, North-Holland Math. Stud., 31, North-Holland, Amsterdam-New York, 1978.

[22] Woldaregay, M. M. Aniley, W. T. and Duressa,G.F. *Novel numerical scheme for singularly perturbed time delay convection-diffusion equation* Adv. Math. Phys. (2021), Art. ID 6641236, 13 pp.

[23] Wu, J. *Theory and applications of partial functional differential equations*, New York, Springer, 119, (2012).

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