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**Research Article**

# Collection-based numerical method for multi-order fractional integro-differential equations

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**Abstract**

In this paper, the standard collocation approach is used to solve multi-order fractional integro-differential equations using Caputo sense. We obtain the integral form of the problem and transform it into a system of linear algebraic equations using standard collocation points. The algebraic equations are then solved using the matrix inversion method. By substituting the algebraic equation solutions into the approximate solution, the numerical result is obtained. We establish the method's uniqueness as well as the convergence of the method. Numerical examples show that the developed method is efficient in problem-solving and competes favorably with the existing method.

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## 1 Introduction

Fractional calculus is one of the subfields of mathematics that looks at the characteristics of the derivatives and integrals of noninteger orders. This discipline examines the notion and method of solving differential equations with fractional derivatives of unknown functions. In recent years, a significant amount of interest in fractional calculus has emerged as a result of the fact that it may be used in a wide variety of fields of scientific interest; see [9]. Some of the numerical methods for the solution of fractional integro-differential equations developed in the literature include: Multi-order fractional by [12, 5, 16], Collocation method by [1, 3], Least square method by [13], Adomian decomposition method by [10], Chebyshev cardinal functions by [8], Laplace decomposition method by [11, 18, 14], Taylor expansion method by [7, 19], Haar wavelets by [4], Legendre Wavelets Method [6], and variational iteration method by [17]. Collocation approach to first-order Volterra integro-differential equations. The class of integro-differential equations was reformulated to assume an approximate solution in terms of the constructed polynomial. After solving for the unknown, we obtained a system of linear algebraic equations by collocating the resulting equation at various places within the range $[0, 1]$ [2]. The Laplace Adomian decomposition technique based on the Bernstein polynomial is employed to obtain an approximate solution for solving Volterra integral and integro-differential equations. Rani and Mistra [15] concluded that only orthogonal polynomials such as Legendre, Chebyshev, or Jacobi polynomials can improve the Adomian decomposition method.

In this research, we present efficient method for solving multi-order fractional integro-differential equations with fractional derivatives of the form

$$D^{\beta}y(x) = \sum_{j=0}^{N} q_j(x)D^{\alpha_j}y(x) + h(x) + \int_0^b k_1(x,t)y(t)dt + \int_0^x k_2(x,t)y(t)dt \tag{1}$$

subject to the initial condition

$$y^{(j)}(a_j) = \lambda_j, \quad j = 0, 1, \ldots, n-1, \; n \in \mathbb{N}, \quad \beta > \alpha_N, \tag{2}$$

where $y(x)$ is the unknown function to be determined, $D^{\alpha_j}$ and $D^{\beta}$ are Caputo's derivative, and $h(x)$ is the force known prior. Moreover, $k_1(x,t)$ and $k_2(x,t)$ are the Fredholm and Volterra integral kernel functions, respectively. Also, $q_j(x)$ is the known function and $a_j$ and $\lambda_j$ are known constants.

## 2 Basic definitions

In this section, we present certain definitions and fundamental ideas of fractional calculus for the purpose of the formulation of the problem that has been presented.

**Definition 1.** The Caputo derivative with order $\alpha > 0$ of the given function $f(x),\ x \in (a, b)$ is defined as [11]

$$ {}_{x}^{C}D_{a}^{\alpha} y(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{x} (x-s)^{m-\alpha-1} y^{(m)}(s) ds, \tag{3} $$

where $m - 1 \leq \alpha \leq m,\ m \in N,\ x > 0$.

**Definition 2.** Let $(a_n), n \geq 0$ be a sequence of real numbers. The power series in $x$ with coefficients $a_n$ is an expression [11]

$$ y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots a_N x^N = \sum_{n=0}^{N} a_n x^n = \phi(x)\ \mathbf{A}, \tag{4} $$

where $\phi(x) = [1 \quad x \quad x^2 \quad \cdots \quad x^N]$, $\mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$. Then $y(x, n) = x^n \mathbf{A},\ n = 0(1)N,\ n \in Z^+$.

**Definition 3** (Standard Collocation Method (SCM))**.** This method is used to determine the desired collocation points within an interval, $[a, b]$ and is given by [1]

$$ x_i = a + \frac{(b-a)i}{N}, \quad i = 1, 2, 3, \ldots, N. \tag{5} $$

**Definition 4.** Let $y(x)$ be a continuous function. Then [3]

$$ {}_{0}I_{x}^{\beta} \left( {}_{0}^{C}D_{x}^{\beta} y(x) \right) = y(x) - \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k, \tag{6} $$

where $m - 1 < \beta < 1$.

**Definition 5.** Let $p(s)$ be an integrable function. Then [3]

$$ {}_{0}I_{x}^{\beta} \left( p(s) \right) = \frac{1}{\Gamma(\beta)} \int_{0}^{x} (x-s)^{\beta-1} p(s) ds. \tag{7} $$

**Definition 6.** The Riemann–Liouville derivative of order $\alpha > 0$ with $n-1 < \alpha < n$ of the power function $f(t) = t^{p-\alpha}$ is given by [11]

$$ D^{\alpha} t^{p} = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}. \tag{8} $$

**Definition 7.** A metric on a set $M$ is a function $d : M \times M \longrightarrow \mathbb{R}$ with the following properties, for all $x, y \in M$ [3],

(a) $d(x, y) \geq 0$,

(b) $d(x, y) = 0 \Longleftrightarrow x = y$,

(c) $d(x, y) = d(y, x)$,

(d) $d(x, y) \leq d(x, z) + d(x, y)$.

If $d$ is a metric on $M$, then the pair $(M, d)$ is called a metric space.

**Definition 8.** Let $(X, d)$ be a metric space. A mapping $T : X \longrightarrow X$ is Lipschitzian if $\exists$ a constant $L > 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$ [3].

## 3 Mathematical background

In this section, we develop an enhanced method for the numerical solution of multi-order fractional integro-differential equations. This method is based on the collocation approach and also considered power series polynomials as our basic function.

**Theorem 1** (Banach's fixed point theorem)**.** Let $(X, d)$ be a complete metric space. It follows that each contraction mapping $T : X \longrightarrow X$ has a unique fixed point $x$ of $T$ in $X$, such that $T(x) = x$.

**Lemma 1** (Integral form)**.** Let $y(x)$ be a solution to (1) subject to (2). Then the integral form is

$$\begin{aligned} y(x) = W(x) &+ \sum_{j=0}^{N} \frac{1}{\Gamma(m_j - \alpha_j)} \frac{1}{\Gamma(\beta)} \\ &\times \int_0^x (x-s)^{\beta-1} q_j(s) \left[ \int_0^s (s-t)^{m_j - \alpha_j - 1} y^{(m_j)}(t) dt \right] ds \\ &+ \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left( \int_0^b k_1(s,t) y(t) dt \right) ds \\ &+ \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left( \int_0^s k_2(s,t) y(t) dt \right) ds, \end{aligned} \tag{9}$$

where

$$W(x) = \sum_{k=0}^{N} \frac{y^{(k)}(0)}{k!} x^k + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} h(s) ds.$$

*Proof.* Multiplying (1) by ${}_0I_x^{\beta}(\cdot)$ gives

$${}_0I_x^{\beta} \left( D^{\beta} y(x) \right) = {}_0I_x^{\beta} \left( \sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x) \right)$$

$$+ \ {}_0I_x^{\beta}\left(h(x)\right) + \ {}_0I_x^{\beta}\left(\int_0^b k_1(x,t)y(t)dt\right) + \ {}_0I_x^{\beta}\left(\int_0^s k_2(s,t)y(t)dt\right). \tag{10}$$

Using (6) on (9) gives

$$y(x) = \sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k + \ {}_0I_x^{\beta}\left(\sum_{j=0}^{N} q_j(x)D^{\alpha_j}y(x)\right) + \ {}_0I_x^{\beta}\left(h(x)\right) + \ {}_0I_x^{\beta}\left(\int_0^b k_1(x,t)y(t)dt\right) + \ {}_0I_x^{\beta}\left(\int_0^s k_2(s,t)y(t)dt\right). \tag{11}$$

Applying (3) and (7) to (11) gives

$$\begin{aligned} y(x) = &\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1} \\ &\times \left(\sum_{j=0}^{N} q_j(x)\frac{1}{\Gamma(m_j-\alpha_j)}\int_0^s (s-t)^{m_j-\alpha_j-1}y^{(m_j)}(t)dt\right)ds \\ &+ \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}h(s)ds \\ &+ \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\left(\int_0^b k_1(x,t)y(t)dt\right)ds \\ &+ \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\left(\int_0^s k_2(s,t)y(t)dt\right)ds. \end{aligned} \tag{12}$$

Substituting (4) into (12) gives

$$\begin{aligned} y(x) = &\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1} \\ &\times \left(\sum_{j=0}^{N} q_j(x)\frac{1}{\Gamma(m_j-\alpha_j)}\int_0^s (s-t)^{m_j-\alpha_j-1}\frac{d^{m_j}}{dt^{m_j}}\left(\phi(t)\right)\ dt\mathbf{A}\right)ds \\ &+ \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}h(s)ds + \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1} \\ &\times \left(\int_0^b k_1(x,t)\phi(t)dt\right)ds\mathbf{A} + \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1} \end{aligned}$$

$$\times\left(\int_0^s k_2(s,t)\phi(t)dt\right)ds\mathbf{A}. \tag{13}$$

□

## 3.1 Method of solution

Collocating at $x_i$ in (13) gives

$$\begin{aligned} y(x_i) = W(x_i) + \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)}\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-s)^{\beta-1}q_j(s) \\ \times\left(\int_0^s (s-t)^{m_j-\alpha_j-1}\frac{d^{m_j}}{dt^{m_j}}\left(\phi(t)\right)\ dt\right)ds\ \mathbf{A} \\ +\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-s)^{\beta-1}\left(\int_0^b k_1(s,t)\phi(t)dt\right)ds\mathbf{A} \\ +\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-s)^{\beta-1}\left(\int_0^s k_2(s,t)\phi(t)dt\right)ds\ \mathbf{A}, \end{aligned} \tag{14}$$

where

$$W(x_i) = \sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x^k + \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}h(s)ds.$$

Simplifying (14) gives

$$\phi(x_i)\mathbf{A} = W(x_i) + \begin{bmatrix} \sum_{j=0}^{N}\frac{1}{\Gamma(m_j-\alpha_j)}\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-s)^{\beta-1}q_j(s) \\ \times\left(\int_0^s (s-t)^{m_j-\alpha_j-1}\frac{d^{m_j}}{dt^{m_j}}\left(\phi(t)\right)dt\right)ds \\ +\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-s)^{\beta-1} \\ \times\left(\int_0^b k_1(s,t)\left(\phi(t)\right)dt + \int_0^s k_2(s,t)\left(\phi(t)\right)dt\right)ds \end{bmatrix}\mathbf{A}. \tag{15}$$

Factorizing the values of $\mathbf{A}$ from (15) gives

$$\begin{bmatrix} \phi(x_i) - \sum_{j=0}^{N}\frac{1}{\Gamma(m_j-\alpha_j)}\frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-s)^{\beta-1}q_j(s) \\ \times\left(\int_0^s (s-t)^{m_j-\alpha_j-1}\frac{d^{m_j}}{dt^{m_j}}\left(\phi(t)\right)dt\right)ds - \\ \frac{1}{\Gamma(\beta)}\int_0^{x_i}(x_i-s)^{\beta-1} \\ \times\left(\int_0^b k_1(s,t)\left(\phi(t)\right)dt + \int_0^s k_2(s,t)\left(\phi(t)\right)dt\right)ds \end{bmatrix}\mathbf{A} = W(x_i). \tag{16}$$

Equation (16) can be in the form

$$V(x_i)\mathbf{A} = W(x_i), \tag{17}$$

where

$$\begin{aligned} V(x_i) = \phi(x_i) - &\sum_{j=0}^{N} \frac{1}{\Gamma(m_j - \alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i - s)^{\beta-1} q_j(s) \\ &\left( \int_0^s (s-t)^{m_j-\alpha_j-1} \frac{d^{m_j}}{dt^{m_j}} (\phi(t)) \; dt \right) ds \; - \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x_i - s)^{\beta-1} \\ &\left( \int_0^b k_1(s,t) (\phi(t)) \, dt + \int_0^s k_2(s,t) (\phi(t)) \, dt \right) ds \end{aligned} \tag{18}$$

and

$$\mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^T$$

multiply both sides of (17) by $V^{-1}(x_i)$ gives

$$\mathbf{A} = V^{-1}(x_i) W(x_i). \tag{19}$$

**Lemma 2.** Let $y(t)$ be approximated by (10) and let

$$L(x) = \; {}_0I_x^{\beta} \left( \sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x) \right). \tag{20}$$

If $q_j(s) = s^{p_j}$, then

$$\mathbf{L}(x;n) = \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} x_i^{\beta+n-\alpha_j+p_j} \mathbf{A}. \tag{21}$$

*Proof.* Applying (3) and (7) into (20) gives

$$\begin{aligned} {}_0I_x^{\beta} \left( \sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x) \right) = &\sum_{j=0}^{N} \frac{1}{\Gamma(m_j - \alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^{x_i} (x-s)^{\beta-1} q_j(s) \\ &\left[ \int_0^s (s-t)^{m_j-\alpha_j-1} y^{(m_j)}(t) dt \right] ds. \end{aligned} \tag{22}$$

Substituting (8) into (22) gives

$${}_0I_x^{\beta} \left( \sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x) \right)$$

$$= \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} S^{p_j}$$
$$\left[\int_0^s (s-t)^{m_j-\alpha_j-1} \left(\frac{\Gamma(n+1)}{\Gamma(n-m_j+1)} t^{n-m_j}\right) dt\right] ds\ \mathbf{A}. \quad (23)$$

Let $s-t=(1-v)s$. Then $t=vs \Longrightarrow \frac{dt}{dv}=s \Longrightarrow dt=sdv$. Substituting them into (23) gives

$$ {}_0I_x^{\beta} \left(\sum_{j=0}^{N} q_j(x) D^{\alpha_j} y(x)\right)$$
$$= \sum_{j=0}^{N} \frac{\Gamma(n+1)}{\Gamma(m_j-\alpha_j)\Gamma(n-m_j+1)} \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} S^{p_j}$$
$$\left[S^{n-\alpha_j} \int_0^1 (1-v)^{m_j-\alpha_j-1} V^{n-m_j} dt\right] ds\ \mathbf{A}. \quad (24)$$

Simplifying (24), we get

$$\mathbf{L}(x;n) = \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)} x^{\beta+n-\alpha_j+p_j} \mathbf{A}. \quad (25)$$

□

**Lemma 3.** Let $y(t)$ be approximated by (10) and let

$$E(x) = {}_0I_x^{\beta} \left[\int_0^b k_1(s,t)y(t)dt + \int_0^s k_2(s,t)y(t)dt\right]. \quad (26)$$

If $k_1(s,t)=s^r t^{\sigma}$ $k_2(s,t)=s^g t^v$, then

$$\mathbf{E}(x;n) = \left(\begin{array}{c} \frac{b^{r+n+1}\Gamma(r+1)}{(\sigma+n+1)\,\Gamma(\beta+r+1)} x^{\beta+r} + \\ \frac{\Gamma(g+v+n+2)}{(v+n+1)\,\Gamma(\beta+g+v+n+2)} x^{\beta+g+v+n+1} \end{array}\right) \mathbf{A}. \quad (27)$$

*Proof.* Applying (10) to (26) gives

$$ {}_0I_x^{\beta} \left[\int_0^b k_1(s,t)y(t)dt + \int_0^s k_2(s,t)y(t)dt\right]$$
$$= \frac{1}{\Gamma(\beta)} \int_0^x (x-s)_0^{\beta-1} \left[\int_0^b k_1(s,t)y(t)dt + \int_0^s k_2(s,t)y(t)dt\right] ds.$$

Substituting $k_1(s,t) = s^r t^\sigma \ k_2(s,t) = s^g t^v$ gives

$$
\begin{aligned}
&{}_0I_x^\beta \left[ \int_0^b k_1(s,t)y(t)dt + \int_0^s k_2(s,t)y(t)dt \right] \\
&= \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left( \int_0^b s^r t^\sigma y(t)dt \right) ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left( \int_0^s s^g t^v y(t)dt \right) ds. \qquad (28)
\end{aligned}
$$

Applying (4) to (28) and simplifying give

$$
\begin{aligned}
&{}_0I_x^\beta \left[ \int_0^b k_1(s,t)y(t)dt + \int_0^s k_2(s,t)y(t)dt \right] \\
&= \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left[ s^r \frac{b^{\sigma+n+1}}{\sigma+n+1} \right] \mathbf{A} \ ds \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left[ s^g \frac{s^{v+n+1}}{v+n+1} \right] \mathbf{A} \ ds. \qquad (29)
\end{aligned}
$$

Let $x - s = (1-u)x$. Then $s = ux \implies ds = xdu$. Substituting them into (29) gives

$$
\begin{aligned}
&{}_0I_x^\beta \left[ \int_0^b k_1(s,t)y(t)dt + \int_0^s k_2(s,t)y(t)dt \right] \\
&= \left( \begin{array}{c} \frac{b^{\sigma+n+1}}{\Gamma(\beta)\ (\sigma+n+1)} \int_0^1 ((1-u)\,x)^{\beta-1} (ux)^r x \ du + \\ \frac{1}{\Gamma(\beta)\ (v+n+1)} \int_0^1 ((1-u)\,x)^{\beta-1} (ux)^{g+v+n+1} \ x \ du \end{array} \right) \mathbf{A}. \qquad (30)
\end{aligned}
$$

Solving (30) gives

$$
\mathbf{E}(x;n) = \left( \begin{array}{c} \frac{b^{\sigma+n+1}\Gamma(r+1)}{(\sigma+n+1)\,\Gamma(\beta+r+1)} x^{\beta+r} + \\ \frac{\Gamma(g+v+n+2)}{(v+n+1)\,\Gamma(\beta+g+v+n+2)} x^{\beta+g+v+n+1} \end{array} \right) \mathbf{A}.
$$

□

**Lemma 4.** Let $y(t)$ be approximated by (10) and let

$$
C(x) = {}_0I_x^\beta \left( h(x) \right). \qquad (31)
$$

If $h(s) = s^m$, then

$$
C(x) = \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)} x^{\beta+m}.
$$

*Proof.* Applying (7) to (31) gives

$$ {}_0I_x^{\beta}\left(h(x)\right) = \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}h(s)\ ds. $$

Substituting for $h(s)$ gives

$$ {}_0I_x^{\beta}\left(h(x)\right) = \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}s^m ds. $$

Let $x-s=(1-u)x,\ s=ux \Longrightarrow \dfrac{ds}{du}=x \Longrightarrow ds = xdu$. Then

$$ C(x) = \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)}x^{\beta+m}. \tag{32} $$

□

**Lemma 5.** Let $y(x)$ be the solution of (1) and (2). Then the numerical result gives

$$ y(x) = \phi(x_i)V^{-1}(x_i)\ W(x_i), \tag{33} $$

where

$$ \begin{aligned} V(x_i) = {} & \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)}x_i^{\beta+n-\alpha_j+p_j} \\ & + \frac{b^{r+n+1}\Gamma(r+1)}{(\sigma+n+1)\,\Gamma(\beta+r+1)}x^{\beta+r} \\ & + \frac{\Gamma(r+\sigma+n+2)}{(\sigma+n+1)\,\Gamma(\beta+r+\sigma+n+2)}x^{\beta+r+\sigma+n+1} \end{aligned} $$

and

$$ W(x_i) = -\sum_{k=0}^{N}\frac{y^{(k)}(0)}{k!}x_i^k + \frac{\Gamma(m+1)}{\Gamma(\beta+m+1)}x_i^{\beta+m}. $$

*Proof.* Approximate solution of (11) is

$$ y(x) = \phi(x)\ \mathbf{A}. $$

From (19) we have $\mathbf{A} = V^{-1}(x_i)\ W(x_i)$ where

$$ \begin{aligned} V(x_i) = {} & \frac{\Gamma(n+1)\Gamma(n-\alpha_j+p_j+1)}{\Gamma(n-\alpha_j+1)\Gamma(\beta+n-\alpha_j+p_j+1)}x_i^{\beta+n-\alpha_j+p_j} \\ & + \frac{b^{r+n+1}\Gamma(r+1)}{(\sigma+n+1)\,\Gamma(\beta+r+1)}x^{\beta+r} \\ & + \frac{\Gamma(r+\sigma+n+2)}{(\sigma+n+1)\,\Gamma(\beta+r+\sigma+n+2)}x^{\beta+r+\sigma+n+1}. \end{aligned} $$

Substituting for **A** in the approximate solution gives the numerical result

$$y(x) = \phi(x_i)V^{-1}(x_i)\ W(x_i).$$

□

## 4 Uniqueness of the solution

In this section, we establish the uniqueness of the method by introducing the following hypothesis:

$$H_1 : q^* = \max_{x\in[0,1]} |q(x)|,$$

$$H_2 : k_1^* = \max_{x\in[0,1]} \int_0^b |k_1(x,t)|\, dt,$$

$$H_3 : k_2^* = \max_{x\in[0,1]} \int_0^x |k_2(x,t)|\, dt,$$

$$H_4 : \left|y_N^{(m_j)} - y^{(m_j)}\right| \leq L_{m_j} |y_N - y|,$$

$$H_5 : u = \max_{x\epsilon j} \sum_{x\epsilon j}^{N} \frac{L_{m_j}}{\Gamma(m_j - \alpha + 1)}.$$

**Lemma 6.** [$q$-contraction] Let $T : X \longrightarrow X$ be a mapping defined by Theorem 1 for $y_1,\ y_2\ \in X$. Then $T$ is $q$-contraction if and only if

$$\frac{1}{\Gamma(\beta+1)}\left[\frac{uq_j^*}{\Gamma(m_j-\alpha_j+1)} + K_1^* + K_2^*\right] < 1.$$

Moreover, there exist a unique solution of $T$.

*Proof.* We have

$$\begin{aligned}(Ty_1)(x) = W(x) + \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} q_j(s) \\ \times \left[\int_0^s (s-t)^{m_j-\alpha_j-1} y_1^{(m_j)}(t)dt\right] ds + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \\ \times \left[\int_0^b k_1(s,t)y_1(t)dt + \int_0^s k_2(s,t)y_1(t)dt\right] ds\end{aligned}$$

and

$$(Ty_2)(x) = W(x) + \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} q_j(s)$$

$$\times \left[\int_0^s (s-t)^{m_j-\alpha_j-1} y_2^{(m_j)}(t)dt\right] ds + \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1}$$
$$\times \left[\int_0^b k_1(s,t) y_2(t)dt + \int_0^s k_2(s,t) y_2(t)dt\right] ds$$

$$\begin{aligned}\left|(Ty_1)(x) - (Ty_2)(x)\right| = &\sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left|q_j(s)\right| \\ &\times \left[\int_0^s (s-t)^{m_j-\alpha_j-1} \left|y_1^{(m_j)}(t) - y_2^{(m_j)}(t)\right| dt\right] ds \\ &+ \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \\ &\times \left[\begin{array}{c} \int_0^b \left|k_1(s,t)\right| \left|y_1(t) - y_2(t)\right| dt \\ + \int_0^s \left|k_2(s,t)\right| \left|(y_1(t) - y_2(t))\right| dt \end{array}\right] ds.\end{aligned}$$

Taking maximum of both sides and using $H_1$ *to* $H_5$ give

$$d\left(Ty_1(x), Ty_2(x)\right) \leq \frac{1}{\Gamma(\beta+1)} \left[\frac{uq_j^*}{\Gamma(m_j-\alpha_j+1)} + K_1^* + K_2^*\right] d(y_N, y).$$

Since $T$ is a contraction,

$$\frac{1}{\Gamma(\beta+1)} \left[\frac{uq_j^*}{\Gamma(m_j-\alpha_j+1)} + K_1^* + K_2^*\right] < 1.$$

□

## 5 Convergence analysis

In this section, we establish the convergence of the method by substituting the approximate solution into (3.0). We have

$$\begin{aligned} y_N(x) = W(x) &+ \sum_{j=0}^{N} \frac{1}{\Gamma(m_j-\alpha_j)} \frac{1}{\Gamma(\beta)} \\ &\times \int_0^x (x-s)^{\beta-1} q_j(s) \left[\int_0^s (s-t)^{m_j-\alpha_j-1} y_N^{(m_j)}(t)dt\right] ds \\ &+ \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left(\int_0^b k_1(s,t) y_N(t)dt\right) ds \\ &+ \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left(\int_0^s k_2(s,t) y_N(t)dt\right) ds. \end{aligned} \tag{34}$$

Subtracting (9) from (34) gives

$$E_N(x) = y_N(x) - y(x).$$

Hence

$$\begin{aligned}&|E_N(x)|\\ &\leq \frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\sum_{j=0}^{N}\frac{1}{\Gamma(m_j-\alpha_j)}q_j(s)\left|\left[\int_0^s (s-t)^{m_j-\alpha_j-1}E_N(t)dt\right]ds\right|\\ &\quad+\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\left[\left|\int_0^b k_1(s,t)\,E_N(t)dt\right|+\left|\int_0^s k_2(s,t)\,E_N(t)dt\right|\right]ds.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\|E_N(x_i)\|_\infty}{\|E_N(t)\|_\infty}\\ &\leq \frac{1}{\Gamma(\beta)}\int_0^{x_i}(x-s)^{\beta-1}\left|\left[\begin{array}{c}\sum_{j=0}^{N}\frac{1}{\Gamma(m_j-\alpha_j)}q_j(s)\left[\int_0^s (s-t)^{m_j-\alpha_j-1}dt\right]\\ +\left[\int_0^b k_1(s,t)\,dt+\int_0^s k_2(s,t)\,dt\right]\end{array}\right]\right|ds.\end{aligned}$$

The method of solution converges.

## 6 Numerical examples

In this section, we present numerical examples to evaluate the effectiveness and clarity of the method. A MAPLE 18 program is used to perform the computations. Let $y_n(x)$ and $y(x)$ be the approximate and exact solutions, respectively. Error $_N = |y_n(x) - y(x)|$.

**Example 1.** [6] Consider the following multi-order Fractional integro-differential equation:

$$D^{1.7}y(x) = x^2D^{1.5}y(x)+xD^{0.5}y(x)-\int_0^x (x-t)\,y(t)dt-\int_0^1 (x+t)\,y(t)dt+f(x)$$

with this condition $y^{'}(0) = y(0) = 0$ and exact solution $y(x) = x^2 + x^3$, and $f(x) = \left(\frac{\Gamma(3)}{\Gamma(1.5)}+\frac{\Gamma(3)}{\Gamma(2.5)}\right)x^{2.5}+\left(\frac{\Gamma(4)}{\Gamma(2.5)}+\frac{\Gamma(4)}{\Gamma(3.5)}\right)x^{3.5}-\frac{\Gamma(3)}{\Gamma(1.3)}x^{0.3}-\frac{\Gamma(4)}{\Gamma(2.3)}x^{1.3}-\frac{x^4}{12}-\frac{x^5}{20}-\frac{7x}{12}-\frac{9}{20}$.

**Solution 1.** Comparing with (1) and (2), $\beta = 1.7, \alpha_1 = 1.5, \alpha_2 = 0.5, k_1(x,t) = (x+t)\,, k_2(x,t) = (x-t)$.

Using $N = 3$ for illustration. Applying (6) gives

$$
\begin{aligned}
y(x) = W(x) &+ \frac{1}{\Gamma(2-1.5)} \frac{1}{\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} s^2 \\
&\left[\int_0^s (s-t)^{2-1.5-1} \frac{\Gamma(n+1)}{\Gamma(n-2+1)} t^{n-2} dt\right] ds\mathbf{A} \\
&+ \frac{1}{\Gamma(1-0.5)} \frac{1}{\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} \\
&s\left[\int_0^s (s-t)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} t^{n-1} dt\right] ds\ \mathbf{A} \\
&- \frac{1}{\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} \\
&\left[\begin{array}{c} \left(x\frac{\Gamma(n+1)}{\Gamma(n+m+1)} 1^{m+n} + \frac{\Gamma(n+1)}{\Gamma(n+m+1)} 1^{m+n}\right) \\ + \left(x\frac{\Gamma(n+1)}{\Gamma(n+m+1)} x^{m+n} - \frac{\Gamma(n+1)}{\Gamma(n+m+1)} x^{m+n}\right) \end{array}\right] ds\ \mathbf{A}, \quad (35)
\end{aligned}
$$

where

$$
W(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} f(s) ds. \quad (36)
$$

Substituting $f(s)$ into (36) gives

$$
\begin{aligned}
W(x) = &\frac{1}{\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} \\
&\left[\begin{array}{c} \left(\frac{\Gamma(3)}{\Gamma(1.5)} + \frac{\Gamma(3)}{\Gamma(2.5)}\right) s^{2.5} + \left(\frac{\Gamma(4)}{\Gamma(2.5)} + \frac{\Gamma(4)}{\Gamma(3.5)}\right) s^{3.5} \\ -\frac{\Gamma(3)}{\Gamma(1.3)} s^{0.3} - \frac{\Gamma(4)}{\Gamma(2.3)} s^{1.3} - \frac{s^4}{12} - \frac{s^5}{20} - \frac{7s}{12} - \frac{9}{20} \end{array}\right] ds. \quad (37)
\end{aligned}
$$

Simplify further

$$
\begin{aligned}
W(x) = &\left(\frac{\Gamma(3)}{\Gamma(1.5)} + \frac{\Gamma(3)}{\Gamma(2.5)}\right) \frac{1}{\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} x^{2.5} ds \\
&+ \left(\frac{\Gamma(4)}{\Gamma(2.5)} + \frac{\Gamma(4)}{\Gamma(3.5)}\right) \frac{1}{\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} x^{3.5} ds \\
&- \frac{\Gamma(3)}{\Gamma(1.3)} \frac{1}{\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} x^{0.3} ds \\
&- \frac{\Gamma(4)}{\Gamma(2.3)} \frac{1}{\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} x^{1.3} ds \\
&- \frac{1}{12\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} x^4 ds - \frac{1}{20\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} x^5 ds \\
&- \frac{7}{12\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} x ds - \frac{9}{20} \frac{1}{\Gamma(1.7)} \int_0^x (x-s)^{1.7-1} x^0 ds
\end{aligned}
$$

$$\begin{aligned}W(x) = &\left(\frac{\Gamma(3)}{\Gamma(1.5)} + \frac{\Gamma(3)}{\Gamma(2.5)}\right)\frac{\Gamma(2.5+1)}{\Gamma(1.7+2.5+1)}x^{1.7+2.5}\\ &\left(\frac{\Gamma(4)}{\Gamma(2.5)} + \frac{\Gamma(4)}{\Gamma(3.5)}\right)\frac{\Gamma(3.5+1)}{\Gamma(1.7+3.5+1)}x^{1.7+3.5}\\ &-\frac{\Gamma(3)}{\Gamma(1.3)}\frac{\Gamma(0.3+1)}{\Gamma(1.7+0.3+1)}x^{1.7+0.3}\\ &-\frac{\Gamma(4)}{\Gamma(2.3)}\frac{\Gamma(1.3+1)}{\Gamma(1.7+1.3+1)}x^{1.7+1.3}\\ &-12\frac{\Gamma(4+1)}{\Gamma(1.7+4+1)}x^{1.7+4} - 20\frac{\Gamma(5+1)}{\Gamma(1.7+5+1)}x^{1.7+5}\\ &-\frac{7\Gamma(1+1)}{12\Gamma(1.7+1+1)}x^{1.7+1} - \frac{9}{20}\frac{\Gamma(0+1)}{\Gamma(1.7+0+1)}x^{1.7+0}. \qquad (38)\end{aligned}$$

Substituting (38) into (35) gives

$$y(x) = \phi(x_i)V^{-1}(x_i)\ W(x_i).$$

We obtain the result

$$y_3 = \begin{pmatrix} 1.8956614056 \times 10^{-10} + 1.4273998650 \times 10^{-12}x \\ +0.9999999968x^2 + 1.0000000024x^3 \end{pmatrix}.$$

Table 1: Exact and approximate values of Example 1

| **x** | **Exact** | **N=3** | **N=4** | **N=6** |
|---|---|---|---|---|
| 0.25 | 0.0781250000 | 0.0781249983 | 0.0781249999 | 0.0781250000 |
| 0.5 | 0.3750000000 | 0.3749999996 | 0.3749999999 | 0.3750000000 |
| 0.75 | 0.9843750000 | 0.9843749992 | 0.9843749995 | 0.9843749999 |
| 1.0 | 2.000000000 | 1.9999999990 | 2.0000000000 | 2.0000000000 |

Table 2: Absolute Error for Example 1

| **x** | $\mathbf{ERR}_3$ | $\mathbf{ERR}_4$ | $\mathbf{ERR}_6$ | $[12]_{64}$ | $[6]_7$ |
|---|---|---|---|---|---|
| 0.25 | 1.7e-9 | 1.0e-10 | 0.0 | 2,45e-4 | 1.000e-5 |
| 0.5 | 4.0e-10 | 1.0e-10 | 0.0 | 1.375e-3 | 1.200e-5 |
| 0.75 | 8.0e-10 | 5.0e-10 | 1.0e-10 | 5.387e-3 | 2.13e-4 |
| 1.0 | 1.0e-9 | 0.0 | 0.0 | 4.166e-3 | 8.970e-4 |

**Example 2.** [5] Consider multi-order Fractional integro-differential equation of the form

$$D^2y(x) = -D^{1.5}y(x) - y(x) + \int_0^1 y(t)dt + x - \frac{1}{2}$$

with the condition $y(0) = y^{'}(0) = 1$, and the exact solution is $y(x) = x + 1$.

**Solution 2.** Comparing with (1) and (2), $\beta = 2, \sim \beta = 1.5, h(x) = x - \frac{1}{2}$.

Use $N = 3$ for illustration.
Write in the integral form

$$\begin{aligned} y(x) = W(x) - \frac{1}{\Gamma(2-1.5)}\frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1} \\ \left[\int_0^s (s-t)^{2-1.5-1}\frac{\Gamma(n+1)}{\Gamma(n-2+1)}t^{n-2}dt\right] ds\ \mathbf{A} \\ - \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}s^n ds\mathbf{A} + \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1} \\ \left[\int_0^1 t^n dt\right] ds\ \mathbf{A} \end{aligned} \tag{39}$$

where

$$W(x) = \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}\left(s - \frac{1}{2}\right) ds \tag{40}$$

$$\begin{aligned} y(x) = W(x) + \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1} \\ \left[\frac{\Gamma(n+1)}{\Gamma(n-0.5)}s^{n+0.5}\right] ds\ \mathbf{A} \\ - \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}s^n ds\mathbf{A} + \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1} \\ \left[\int_0^1 t^n dt\right] ds\ \mathbf{A} \end{aligned}$$

$$\begin{aligned} y(x) = W(x) + \frac{\Gamma(n+1)\Gamma(n+1.5)}{\Gamma(n-0.5)\Gamma(n+3.5)}x^{n+4.5}ds\ \mathbf{A} \\ - \frac{\Gamma(n+1)}{\Gamma(n+3)}x^{n+4}\mathbf{A} + \frac{\Gamma(n+1)}{\Gamma(n+4)}1^{n+5}\ \mathbf{A} \end{aligned} \tag{41}$$

$$W(x) = \frac{\Gamma(2)}{\Gamma(4)}x^5 - \frac{\Gamma(1)}{2\Gamma(3)}x^4 \tag{42}$$

for $n = 0(1)N$. Applying (41) and (42) gives

$$y(x) = \phi(x_i)V^{-1}(x_i)\ W(x_i).$$

we obtain the result

$$y_3(x) = \begin{pmatrix} 1.0000000000 + 1.0000000000x+ \\ 8.8817841970 \times 10^{-16}x^2 + 2.2204460493 \times 10^{-16}x^3 \end{pmatrix}.$$

Table 3: Exact and approximate values of Example 2

| **x** | **Exact** | **N=3** | **Absolute Error** |
|---|---|---|---|
| 0.2 | 1.2000000000 | 1.2000000000 | 0.00 |
| 0.4 | 1.4000000000 | 1.4000000000 | 0.00 |
| 0.6 | 1.6000000000 | 1.6000000000 | 0.00 |
| 0.8 | 1.8000000000 | 1.8000000000 | 0.00 |
| 1.0 | 2.0000000000 | 2.0000000000 | 0.00 |

**Example 3.** [5] consider fractional fredholm integro-differential equations of the form

$$D^{1.5}y(x) = D^{0.5}y(x) + \int_0^1 e^x y(t)dt + f(x),$$

where $f(x) = e^x - e^{x+1}$ with the condition $y(0) = 0$ and exact solution $y(x) = e^x$.

**Solution 3.** Comparing with (1) and (2), we have $\beta = 1.5, \alpha = 0.5, \sim k(x,t) = e^x, \sim f(x) = e^x - e^{x+1}$.

Write in the integral form

$$\begin{aligned} y(x) = W(x) + \sum_{j=0}^{N} \frac{1}{\Gamma(m_j - \alpha_j)} \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \\ \left[\int_0^s (s-t)^{m_j-\alpha_j-1} \frac{\Gamma(n+1)}{\Gamma(n-m_j+1)} t^{n-m_j} dt\right] ds \ \mathbf{A} \\ - \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} \left[\int_0^1 e^s \ t^n dt\right] ds \ \mathbf{A}, \end{aligned} \tag{43}$$

where

$$W(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} f(s) ds. \tag{44}$$

Use $N = 3$ for illustration.

Substituting for $\beta = 1.5, \alpha = 0.5, f(x) = e^x - e^{x+1}$ in (43) and (44) gives

$$\begin{aligned} y(x) = W(x) + \frac{1}{\Gamma(1-0.5)} \frac{1}{\Gamma(1.5)} \int_0^x (x-s)^{1.5-1} \\ \left[\int_0^s (s-t)^{1-0.5-1} \frac{\Gamma(n+1)}{\Gamma(n-1+1)} t^{n-1} dt\right] ds \ \mathbf{A} \end{aligned}$$

$$+\frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{1.5-1}\left[\int_0^1 e^s\ t^n dt\right] ds\ \mathbf{A}$$

$$W(x)=\frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{1.5-1}\left(e^s-e^{s+1}\right) ds$$

for $n=0(1)N$. Applying Lemma 4 gives

$$\begin{aligned} y(x)&=W(x)+\frac{\Gamma(n+1)}{\Gamma(n+0.5)}\frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{1.5-1}s^{n+1.5}ds\ \mathbf{A} \\ &+\frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{1.5-1}\left[\int_0^1 e^s\ t^n dt\right] ds\ \mathbf{A}, \end{aligned} \tag{45}$$

where

$$\begin{aligned} W(x)&=\frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{1.5-1}\frac{s^n}{\Gamma(n+1)}ds \\ &-\frac{1}{\Gamma(1.5)}\int_0^x (x-s)^{1.5-1}\frac{(s+1)^n}{\Gamma(n+1)}ds. \end{aligned} \tag{46}$$

Using (45) and (46) gives

$$y(x)=\phi(x_i)V^{-1}(x_i)\ W(x_i).$$

We obtain the result

$$y_3=\begin{pmatrix} 0.9990233401+1.0116982759x+ \\ 0.4050677749x^2+0.3003042742x^3 \end{pmatrix}.$$

Table 4: Exact and approximate values of Example 3

| x | Exact | N=3 | N=5 | N=6 |
|---|---|---|---|---|
| 0.2 | 1.2214027580 | 1.2199681400 | 1.2213960720 | 1.2214031090 |
| 0.4 | 1.4918246980 | 1.4877329680 | 1.4918178290 | 1.4918249590 |
| 0.6 | 1.8221188000 | 1.8167324280 | 1.8221150110 | 1.8221190520 |
| 0.8 | 2.2255409280 | 2.2213811250 | 2.2255348760 | 2.2255409420 |
| 1.0 | 2.7182818280 | 2.7160936650 | 2.7182828500 | 2.7182817820 |

**Example 4.** [16] consider the initial value problem of equation

$$D^2y(x)=x^2D^{1.5}y(x)+x^{\frac{1}{2}}D^{1.5}y(x)+x^{\frac{1}{3}}y(x)+f(x), \quad 0<x\le 1,$$

$$y(0)=0,\ y^{'}(0)=0$$

Table 5: Absolute error for Example 3

| x | $\mathbf{ERR_3}$ | $\mathbf{ERR_5}$ | $\mathbf{ERR_6}$ | $[5]_{\mathbf{N=18}}$ |
|---|---|---|---|---|
| 0.2 | 1.4346180e-3 | 6.686e-6 | 3.51e-7 | 0.21823e-7 |
| 0.4 | 4.09173e-3 | 6.869e-6 | 2.61e-7 | 0.21586e-7 |
| 0.6 | 5.386372e-3 | 3.789e-6 | 2.52e-7 | 0.86325e-7 |
| 0.8 | 4159803e-3 | 6.052e-6 | 1.40e-8 | 0.12423-5 |
| 1.0 | 2.1881634e-3 | 1.022e-6 | 4.60e-8 | 0.83792e-5 |

$$f(x) = 6\pi^{1/2} - 8x^{7/2} - \frac{16}{5}x^3 - x^{10/3}\pi^{1/2}.$$

The exact solution is

$$y(x) = \pi^{1/2}x^3.$$

**Solution 4.** Comparing with (1) and (2), we have $\beta = 2, \alpha_1 = 1.5, \alpha_2 = 0.5, h(x) = 6\pi^{1/2} - 8x^{7/2} - \frac{16}{5}x^3 - x^{10/3}\pi^{1/2} \sim$.

Use $N = 3$ for illustration. Applying (6) gives

$$\begin{aligned} y(x) = W(x) &+ \frac{1}{\Gamma(2-1.5)}\frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}\ s^2 \\ &\left[\int_0^s (s-t)^{2-1.5-1}\frac{\Gamma(n+1)}{\Gamma(n-2+1)}t^{n-2}dt\right] ds\ \mathbf{A} \\ &+\frac{1}{\Gamma(1-0.5)}\frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}\ s^{1/2} \\ &\left[\int_0^s (s-t)^{1-0.5-1}\frac{\Gamma(n+1)}{\Gamma(n-1+1)}t^{n-1}dt\right] ds\ \mathbf{A} \\ &+ \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}\ s^{1/3}\left[seq(s^n, n=0,\ldots,N)\right] ds\ \mathbf{A}, \end{aligned} \tag{47}$$

for $n = 0(1)N$, where

$$W(x) = \frac{1}{\Gamma(2)}\int_0^x (x-s)^{2-1}\left(6\pi^{1/2}s - 8s^{7/2} - \frac{16}{5}s^3 - s^{10/3}\pi^{1/2}\right) ds.$$

Simplifying (47) gives

$$\begin{aligned} y(x) = W(x) &+ \frac{\Gamma(n+1)\Gamma(n-1)\Gamma(3.5+n)}{\Gamma(n-3)\Gamma(n-0.5)\Gamma(5.5+n)}x^{6.5+n}\mathbf{A} \\ &+ \frac{\Gamma(n-1)}{\Gamma(n+0.5)}x^{n+2}\ \mathbf{A} + \frac{\Gamma(n+0.8)}{\Gamma(n+2.8)}x^{n+3.8}\ \mathbf{A}, \end{aligned} \tag{48}$$

$$W(x) = \frac{6\pi^{1/2}\Gamma(2)}{\Gamma(4)}x^5 - \frac{8\Gamma(3.5)}{\Gamma(5.5)}x^{6.5} - \frac{16}{5}\frac{\Gamma(3)}{\Gamma(5)}x^6 - \frac{\pi^{1/2}\Gamma(3.3)}{\Gamma(5.3)}x^{6.3} \tag{49}$$

Substituting (49) into (48) gives

$$y(x) = \phi(x_i)V^{-1}(x_i)\ W(x_i).$$

We obtain the result

$$y_3 = \begin{pmatrix} -0.778452e - 4x^0 - 0.212739e - 4x + \\ 0.12350293e - 2x^2 + 1.7678381513x^3 \end{pmatrix}.$$

Table 6: Exact and approximate values of Example 4

| x | Exact | N=3 | N=5 | N=7 | N=10 |
|---|---|---|---|---|---|
| 0.1 | 0.0017725688 | 0.0017002159 | 0.0017710059 | 0.0017729787 | 0.0017721932 |
| 0.3 | 0.0478593564 | 0.0477585554 | 0.0478450493 | 0.0478592986 | 0.0478593029 |
| 0.5 | 0.2215710946 | 0.2212000441 | 0.2215300075 | 0.2215712791 | 0.2215710855 |
| 0.7 | 0.6079910837 | 0.6068809132 | 0.6079068083 | 0.6079913542 | 0.6079910989 |
| 0.9 | 1.2922026240 | 1.2896573940 | 1.2920556370 | 1.2922025970 | 1.2922026170 |

Table 7: Absolute error for Example 4

| x | $\mathbf{ERR}_3$ | $\mathbf{ERR}_5$ | $\mathbf{ERR}_7$ | $\mathbf{ERR}_{10}$ | $[16]_{10}$ |
|---|---|---|---|---|---|
| 0.1 | 7.23529e-5 | 1.5629e-6 | 4.099e-7 | 3.756e-7 | 7.45873e-7 |
| 0.3 | 1.00801e-4 | 1.43072e-5 | 4.422e-6 | 5.35e-7 | 1.4833e-6 |
| 0.5 | 3.7105e-4 | 4.1087e-5 | 1.845e-7 | 9.1e-9 | 1.74701e-6 |
| 0.7 | 1.1101e-3 | 8.4275e-5 | 1.705e-7 | 1.52e-9 | 5.5116e-7 |
| 0.9 | 2.54523e-3 | 1.46987e-4 | 2.7e-8 | 7.0e-9 | 2.47276e-6 |

## 7 Discussion of results

In this section, we discuss the numerical results obtained from the solved examples using the derived numerical method.

In Example 1, the approximate solution obtained as $N = 3$ gives $y_3 = 1.8956614056 \times 10^{-10} + 1.4273998650 \times 10^{-12}x + 0.9999999968x^2 + 1.0000000024x^3$. Solving for $N = 4$ and $N = 6$, we obtained Table 1, which shows the results obtained from solving Example 1. Table 2 shows the absolute error of Example 1, and it indicates that as the values of $N$ increase, the error becomes smaller and more consistent across all values of $x$. For instance, the least error of [12] at $N = 64$ is $2.45e - 4$ while the least error in our method is 0.00 at $N = 6$. This confirmed that our method performed better.

In Example 2, the approximate solution obtained at $N = 3$ gives $y_3(x) = 1.00000000 + 1.0000000000x + 8.8817841970 \times 10^{-16}x^2 + 2.2204460493 \times 10^{-16}x^3$, which shows that the result converges to the exact solution as displayed in Table 3.

In Example 3, the approximate solution at $N = 3$ gives $y_3(x) = 0.9990233401 + 1.0116982759x + 0.4050677749x^2 + 0.3003042742x^3$. Solving $N = 5$ and 7, we obtained Table 4, which displays the results obtained at $x = 0.2$ *to* 1.0 for various values of $N$ and the exact solution. The absolute error of Example 3 as shown in Table 5 indicates that as the values of $N$ increase, the error becomes smaller. For instance, the least error in [5] at $N = 18$ is $0.21823e - 7$ while the least error in our method at $N = 6$ is $1.40e - 8$. This shows that the numerical method developed is consistent and converges faster.

In Example 4, the approximate solution at $N = 3$ gives $y_3(x) = -2.5324187192 \times 10^{-13} + 6.5978333907 \times 10^{-12}x - 1.0000000002x^2 + 1.0000000000x^3$. Solving at $N = 5$, $N = 7$, and $N = 10$, we obtained Table 6, which shows the results obtained at $x = 0.1$ *to* 0.9 for various values of $N$ and the exact solution. Table 7 shows the absolute error of problem 1, and it indicates that as the value of $N$ increases, the error becomes smaller. We also compare our results with [16]. For instance, the least error in [16]at $N = 10$ is $5.5116e - 7$ while the least error in our method is $2.7e - 8$ at $N = 7$. This clearly shows that our method performs better.

Hence, from the numerical results obtained, we can conclude that the numerical method derived is efficient, consistent, and computationally reliable.

## 8 Conclusion

An enhanced numerical method was developed for the solution of multi-order fractional integro-differential equations with initial conditions using the collocation method. The numerical method derived is consistent, efficient, and reliable. Maple code was used to implement the developed method. Solved numerical examples showed that the method is reliable and suitable for such kinds of problems.

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