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GORENSTEIN n-WEAK INJECTIVE AND GORENSTEIN n-WEAK FLAT MODULES UNDER EXCELLENT EXTENSIONS

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Abstract. Let R be an associative ring and let n be a non-negative integer. In this paper, we consider n-super finitely presented, Gorenstein n-weak injective and Gorenstein n-weak flat modules under change of rings. For an excellent extension $S \geq R$, we show that the Gorenstein n-super finitely presented dimensions (resp., weak Gorenstein n-super finitely presented dimensions) of rings R and S coincide.

Keywords: Almost excellent extension, Excellent extension, Gorenstein n-weak injective module, Gorenstein n-weak flat module, n-super finitely presented module.
2020 MSC: Primary 16D80, 16E05, 16E30, 16E65, 16P70.

1. Introduction

In 2015, Gao and Wang [5] introduced the notion of weak injective and weak flat modules using super finitely presented modules. Then in 2020, Zhao and Xu [14] using these modules introduced the notion of Gorenstein weak injective and Gorenstein weak flat modules, and then some functions on arbitrary rings were investigated. The category of these modules are smaller than the category of Gorenstein weak injective and weak flat modules. In recent years, the homological theory of these modules has become an important research field, (see [1–5, 11, 13, 14]).

Recently, in 2021, Amini et al., [1], via n-super finitely presented modules introduced stronger concepts than weak injective and weak flat modules called n-weak injective and n-weak flat modules. Then, in 2023 in [2], with the help of special super finitely presented modules of every n-super finitely presented module and also n-weak injective modules and n-weak flat modules, they introduced the notion of Gorenstein n-weak injective and Gorenstein n-weak flat modules. They presented several examples of these modules, and then, they investigateded some homological functions on arbitrary rings. We show the classes of finitely generated, n-finitely presented, n-super finitely presented, n-weak injective, Gorenstein n-weak injective left R-modules

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and n-weak flat, Gorenstein n-weak flat right R-modules by $FG(R)$, $F^nP(R)$, $S^nFP(R)$, $W^nI(R)$, $GW^nI(R)$, $W^nF(R^{op})$ and $GW^nF(R^{op})$, respectively. If $n = 0$, then $F^nP(R) = FP(R)$, $S^nFP(R) = SFP(R)$, $W^nI(R) = WI(R)$, $GW^nI(R) = GWI(R)$, $W^nF(R^{op}) = WF(R^{op})$ and $GW^nF(R^{op}) = GWF(R^{op})$ are classes of super finitely presented, weak injective, Gorenstein weak injective left R-modules and weak flat, Gorenstein weak flat right R-modules, respetively.

This paper is devoted to study homological behavior of classes $S^nFP(R)$, $W^nI(R)$, $GW^nI(R)$, $W^nF(R^{op})$ and $GW^nF(R^{op})$ with respect to change of rings. The obtained results will be used in homological algebra.

The article is planned in the following order:

Section 2 is devoted to some preliminary concepts and results. In Section 3, we examine and study some properties of classes $S^nFP(R)$, $W^nI(R)$ and $W^nF(R^{op})$ with respect to change of rings. For instance, for an excellent extension $S \geq R$, we show that (1) $M \in W^nI(R)$ if and only if $\mathrm{Hom}_R(S, M) \in W^nI(S)$. (2) $M \in W^nF(R^{op})$ if and only if $M \otimes_R S \in W^nF(S^{op})$. (3) Let $n \geq 0$. Then l.n.sp.gl.idim(R) = l.n.sp.gl.idim(S), where l.n.sp.gl.idim(R) is the n-super finitely presented dimension of ring R.

In Section 4, we study some results of classes $GW^nI(R)$ and $GW^nF(R^{op})$ with respect to change of rings. For instance, we consider an excellent extension $S \geq R$. Then we show (1) $M \in GW^nI(R)$ if and only if $\mathrm{Hom}_R(S, M) \in GW^nI(S)$, (2) $M \in GW^nF(R^{op})$ if and only if $M \otimes_R S \in GW^nF(S^{op})$, (3) Let $n \geq 0$. Then l.G.n.sp.gl.idim(R) = l.G.n.sp.gl.idim(S), where l.G.n.sp.gl.idim(R) is the Gorenstein n-super finitely presented dimension of ring R, and (4) If $n \geq 0$, then r.G.n.sp.gl.fdim(R) = r.G.n.sp.gl.fdim(S), where r.G.n.sp.gl.fdim(R) is the weak Gorenstein n-super finitely presented dimension of ring R.

2. **Preliminaries**

Throughout this paper R will be an associative ring with identity, and all modules will be unital. Section 2 is devoted to some basic concepts and some preliminary results.

A left R-module M is said to be *finitely n-presented* [12] ($M \in F^nP(R)$) if an exact sequence

$$Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

of projective R-modules $Q_i \in FG(R)$ exists.

A left R-module U is called *super finitely presente* [4] ($U \in SFP(R)$) if an exact sequence

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow U \longrightarrow 0$$

of projective R-modules $Q_i \in FG(R)$ exists. A left R-module M is called *weak injective* [5] ($M \in WI(R)$) if $\mathrm{Ext}^1_R(U, M) = 0$ for each $U \in SFP(R)$, and a right R-module M is called *weak flat* ($M \in WF(R^{op})$) if $\mathrm{Tor}^R_1(M, U) = 0$

for each $U \in SFP(R)$. A left R-module M is said to be *Gorenstein weak injective* [14] ($M \in GWI(R)$) if an exact sequence of the form:

$$\mathbf{W} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots,$$

where $W_i, W^i \in WI(R)$ exists, with $M = \mathrm{Coker}(W_1 \to W_0)$ and $\mathrm{Hom}_R(N, \mathbf{W})$ is exact with respect to every $N \in SFP(R)$ with $\mathrm{pd}_R(N) < \infty$. A right R-module M is said to be *Gorenstein weak flat* [14] ($M \in GWF(R^{op})$) if an exact sequence of the form:

$$\mathbf{W} = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots,$$

where $W_i, W^i \in WF(R^{op})$ exists, with $M = \mathrm{Coker}(W_1 \to W_0)$ and $\mathbf{W} \otimes_R N$ is exact with respect to every $N \in SFP(R)$ with $\mathrm{pd}_R(N) < \infty$.

Definition 2.1. [1, Definition 2.1] Let $n \geq 0$. Then a left R-module U is said to be *n-super finitely presented* ($U \in S^nFP(R)$) if there is an exact sequence of projective R-modules Q_i of the form:

$$\cdots \to Q_{n+1} \to Q_n \to \cdots \to Q_1 \to Q_0 \to U \to 0,$$

where for any $i \geq n$, $Q_i \in FG(R)$. Let $K_i := \mathrm{Im}(Q_{i+1} \to Q_i)$. Then for $i = n-1$, the module K_{n-1} is said to be *special super finitely presented*. We denote the class of special super finitely presented left R-modules by $SSFP(R)$.

Let $K_{n-1} \in SSFP(R)$ with respect to any $U \in S^nFP(R)$. Then $\mathrm{Ext}_R^{n+1}(U, -) \cong \mathrm{Ext}_R^1(K_{n-1}, -)$ and $\mathrm{Tor}_{n+1}^R(-, U) \cong \mathrm{Tor}_1^R(-, K_{n-1})$. For any $m \geq n$, $S^nFP(R) \subseteq S^mFP(R)$, but not conversely (see Examples 2.2).

The *finitely presented dimension of an R-module A* is defined as: $\mathrm{f.p.dim}_R(A) = \inf\{n \mid$ there exists an exact sequence $\mathrm{Q}_{n+1} \to \mathrm{Q}_n \to \cdots \to \mathrm{Q}_0 \to \mathrm{A} \to 0$ of projective R-modules, where $\mathrm{Q}_n, \mathrm{Q}_{n+1} \in FG(R)\}$. The *finitely presented dimension of ring R* is defined as:

$$\mathrm{f.p.dim}(R) = \sup\{\mathrm{f.p.dim}_R(A) \mid A \in FG(R)\}.$$

The *global dimension of ring R* is defined as: $\mathrm{gl.dim}(R) = \sup\{\mathrm{i.dim}_R(A) \mid A \text{ is a left } R\text{-module}\} = \sup\{\mathrm{p.dim}_R(A) \mid A \text{ is a left } R\text{-module}\}$.

The *weak global dimension of ring R* is defined as: $\mathrm{w.gl.dim}(R) = \sup\{\mathrm{f.dim}_R(A) \mid A \text{ is a right } R\text{-module}\}$.

Also, R is called an *(a,b,c)-ring* if $\mathrm{w.gl.dim}(R) = a$, $\mathrm{gl.dim}(R) = b$ and $\mathrm{f.p.dim}(R) = c$ (see [8]).

Example 2.2. Let $k[[x_1, x_2, x_3, x_4]]$ be the ring of power series in 4 indeterminates over a field k, and let S be a valuation ring with global dimension 4. Then by [8, Proposition 3.10], $R = k[[x_1, x_2, x_3, x_4]] \oplus S$ is $(4,4,5)$-ring and coherent. So $\mathrm{f.p.dim}(R) = 5$, hence $\mathrm{f.p.dim}_R(U) = 5$ for some $U \in FG(R)$. So there is an exact sequence

$$Q_6 \longrightarrow Q_5 \longrightarrow Q_4 \longrightarrow Q_3 \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow U \longrightarrow 0,$$

where $Q_5, Q_6 \in FG(R)$ and every Q_i is projective. R is coherent, hence $K_4 := \text{Im}(Q_5 \to Q_4) \in SSFP(R)$. So by Definition 2.1, $U \in S^5FP(R)$. But $U \not inS^4FP(R)$ otherwise $\text{f.p.dim}_R(U) = 4$, a contradiction.

Definition 2.3. [1, Definition 2.2] Let $n \geq 0$. Then a left R-module M is called *n-weak injective* ($M \in W^nI(R)$) if $\text{Ext}_R^{n+1}(U, M) = 0$, for every $U \in S^nFP(R)$. A right R-module N is called *n-weak flat* ($N \in W^nF(R^{op})$) if $\text{Tor}_{n+1}^R(N, U) = 0$, for every $U \in S^nFP(R)$.

Example 2.4. Let $k[[x_1, x_2, x_3]]$ be the ring of power series in 3 indeterminates over a field k, and let S be a valuation ring of global dimension 2. Then by [8, Proposition 3.8], $R = k[[x_1, x_2, x_3]] \oplus S$ is a coherent $(3, 3, 3)$-ring. Hence for every R-module M, $M \in W^3I(R)$, since $\text{gl.dim}(R) = 3$. But there exists an R-module M that $M \notin WI(R)$. Indeed, if $M \in WI(R)$ for every module M, then $\text{Ext}_R^1(U', M) = 0$ for any $U' \in SFP(R)$. So U' is projective. We know that $FP(R) \subseteq SFP(R)$, since R is coherent. So every module in $FP(R)$ is projective. Hence R is regular, and thus all modules are flat. Therefore $\text{w.gl.dim}(R) = 0$, a contradiction. Similarly, from $\text{w.gl.dim}(R) = 3$, we get that for every R-module M, $M \in W^3F(R)$, but there is an R-module M that $M \notin WF(R)$.

Let $n \geq 0$. Then by [2, Corollaries 3.7 and 3.8], an R-module M is *Gorenstein n-weak injective* ($M \in GW^nI(R)$) if and only if there exists an exact sequence

$$\mathbf{WI} = \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots,$$

where $X^i, X_i \in W^nI(R)$ and $M = \text{Ker}(X^0 \to X^1)$. Also, a right R-module N is *Gorenstein n-weak flat* ($N \in GW^nF(R^{op})$) if and only if there exists the following exact sequence

$$\mathbf{WF} = \cdots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow \cdots,$$

where $Y^i, Y_i \in W^nF(R^{op})$ and $N = \text{Ker}(Y^0 \to Y^1)$.
It is clear that $W^nI(R) \subseteq GW^nI(R)$ and $W^nF(R^{op}) \subseteq GW^nF(R^{op})$, because if $X \in W^nI(R)$ (resp. $X \in W^nF(R^{op})$), then there is an exact sequence $\mathbf{WI} = \cdots \to X \to X \to X \to X \to \cdots$, where $X = \text{Ker}(X \to X)$.

3. n-weak injective and n-weak flat modules under change of rings

This section investigates the classes $S^nFP(R)$, $W^nI(R)$ and $W^nF(R^{op})$ and also dimensions under change of rings.

The following lemma consider that the behaviour of the classes $S^nFP(R)$ in short exact sequences is the same as the one of the classical homological notions.

Lemma 3.1. *Let $n \geq 1$ and $0 \to A \to B \to C \to 0$ be an exact sequence of left R-modules. Then*

(1) *If $B \in S^nFP(R)$ and $A \in S^{n-1}FP(R)$, then $C \in S^nFP(R)$.*
(2) *If $C \in S^nFP(R)$ and $B \in S^{n-1}FP(R)$, then $A \in S^{n-1}FP(R)$.*
(3) *If $A, C \in S^{n-1}FP(R)$, then $B \in S^{n-1}FP(R)$.*

Proof. (1) By hypothesis, $A \in S^{n-1}FP(R)$. So, there is an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0,$$

where every P_i is projective and for any $i \geq n-1$, $P_i \in FG(R)$. Also for $B \in S^nFP(R)$, there is an exact sequence

$$\cdots \to Q_{n+1} \to Q_n \to \cdots \to Q_1 \to Q_0 \to B \to 0,$$

where every Q_i is projective and for any $i \geq n$, $Q_i \in FG(R)$. Therefore by [7, Theorem 3.6], there is exact sequence

$$\cdots \to Q_{n+1} \oplus P_n \to Q_n \oplus P_{n-1} \to \cdots \to Q_1 \oplus P_0 \to Q_0 \to C \to 0$$

of projective R-modules exists, where $Q_i \oplus P_{i-1} \in FG(R)$ for any $i \geq n$. Hence, it follows that $C \in S^nFP(R)$.

(2) Since $B \in S^{n-1}FP(R)$, there is an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to B \to 0,$$

where every P_i is projective and for any for any $i \geq n-1$, $P_i \in FG(R)$. Also, there is a exact sequence

$$\cdots \to Q_{n+1} \to Q_n \to \cdots \to Q_1 \to Q_0 \to C \to 0$$

of projective R-modules Qi, where $Q_i \in FG(R)$ for any $i \geq n$. Hence by [7, Theorem 3.2], there exist exact sequences

$$\cdots \to Q_{n+1} \oplus P_n \to Q_n \oplus P_{n-1} \to \cdots \to Q_2 \oplus P_1 \to P \to A \to 0$$

and

$$0 \to P \to Q_1 \oplus P_0 \to Q_0 \to 0$$

of projective R-modules, where $Q_i \oplus P_{i-1} \in FG(R)$ for any $i \geq n$. Hence $A \in S^{n-1}FP(R)$.

(3) Since $A, C \in S^{n-1}FP(R)$, there exist projective resolutions of A and C as

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

and

$$\cdots \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to C \to 0,$$

where $P_i, Q_i \in FG(R)$ for any $i \geq n-1$. So by [9, Proposition 6.24], we have the sequence

$$\cdots \to Q_n \oplus P_n \to Q_{n-1} \oplus P_{n-1} \to \cdots \to Q_1 \oplus P_1 \to Q_0 \oplus P_0 \to B \to 0$$

of projective R-modules, where $Q_i \oplus P_i \in FG(R)$ for any $i \geq n-1$, and thus $B \in S^{n-1}FP(R)$. □

We assume $S \geq R$ is a unitary ring extension. The ring S is called *right R-projective* [12], in case for any right S-module M_S with an S-submodule N_S, $N_R \mid M_R$ implies $N_S \mid M_S$, where $N \mid M$ means N is a direct summand of M. S is called a *finite normalizing extension* of R if there exist elements $a_1, \ldots, a_n \in S$ such that $S = Ra_1 + \cdots + Ra_n$ and $a_iR = Ra_i$ for $i = 1, \ldots, n$. A finite normalizing extension $S \leq R$ is called an *almost excellent extension* in case ${}_RS$ is flat, S_R is projective, and the ring S is right R-projective. An almost excellent extension $S \geq R$ is an *excellent extension* in case both ${}_RS$ and S_R are free modules with a common basis $\{a_1, \ldots, a_n\}$, the class of excellent extensions includes finite matrix rings, and crossed product R^*G where G is a finite group with $\mid G \mid^{-1} \in R$, (see [12]).

Lemma 3.2. *Consider an almost excellent extension $S \geq R$. If $U \in S^nFP(R)$, then $S \otimes_R U \in S^nFP(S)$.*

Proof. Let $U \in S^nFP(R)$. Then there is an exact sequence

$$\cdots \longrightarrow P_{n+1} \cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow U \longrightarrow 0,$$

where every P_i is projective and for any $i \geq n$, $P_i \in FG(R)$. Since S_R is flat, the sequence

$$\cdots \longrightarrow S \otimes_R P_{n+1} \cdots \longrightarrow S \otimes_R P_n \longrightarrow \cdots \longrightarrow S \otimes_R P_0 \longrightarrow S \otimes_R U \longrightarrow 0,$$

is exact, where every $S \otimes_R P_i$ is projective, and also $S \otimes_R P_i \in FG(S)$ for any $i \geq n$. □

Lemma 3.3. *Consider an almost excellent extension $S \geq R$. Then for a left S-module U, ${}_SU \in S^nFP(S)$ if and only if ${}_RU \in S^nFP(R)$.*

Proof. ($\Longrightarrow$) We use induction on n. Let $n = 0$, then $U \in SFP(S)$. So, there is exact sequence

$$\cdots \longrightarrow {}_SP_m \longrightarrow \cdots \longrightarrow {}_SP_3 \longrightarrow {}_SP_2 \longrightarrow {}_SP_1 \longrightarrow {}_SP_0 \longrightarrow {}_SU \longrightarrow 0,$$

of projective S-modules, where ${}_SP_i \in FG(S)$ for any $i \geq 0$. Also, the exact sequence $0 \to {}_SK_0 \to {}_SP_0 \to {}_SU \to 0$ exists, where ${}_SK_0$ is $(m-1)$-presented for any $m \geq 1$. Then by [12, Theorem 1], ${}_SU$ is m-presented for any $m \geq 0$.

Hence by [12, Theorem 5], we deduce that ${}_RU$ is m-presented for any $m \geq 0$, and so we obtain that $U \in SFP(R)$.

Now, let $n \geq 1$. Consider, the exact sequence $0 \to {}_SK_0 \to {}_SP_0 \to {}_SU \to 0$. Since $U \in S^nFP(S)$, by Lemma 3.1, $K_0 \in S^{n-1}FP(S)$. So by induction hypothesis, $K_0 \in S^{n-1}FP(R)$. Also, by [12, Theorem 5], ${}_RP_0 \in FG(R)$. On the other hand, $\mathrm{Ext}^1_R(P_0, N) \cong \mathrm{Ext}^1_R(P_0 \otimes_S S, N) \cong \mathrm{Ext}^1_S(P_0, \mathrm{Hom}_R(S, N)) = 0$, for every R-module N, and hence ${}_RP_0$ is projective. So by Lemma 3.1, $U \in S^nFP(R)$.

($\Longleftarrow$) Let $U \in S^nFP(R)$. Then by Lemma 3.2, $S \otimes_R U \in S^nFP(S)$. Also, similar to the proof of [12, Theorem 5], ${}_SU$ is isomorphic to a summand of $S \otimes_R U$. So we get taht $U \in S^nFP(S)$, too. □

For any R-module M, $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ denotes the character module of M.

Proposition 3.4. *Consider an almost excellent extension $S \geq R$. Then*

(1) *If $M \in W^nI(R)$, then $\mathrm{Hom}_R(S, M) \in W^nI(S)$.*
(2) *If $M \in W^nF(R^{op})$, then $M \otimes_R S \in W^nF(S^{op})$.*

Proof. (1) Let $U \in S^nFP(S)$. Then by Lemma 3.3, $U \in S^nFP(R)$, too. Note that

$$\mathrm{Ext}^{n+1}_S(U, \mathrm{Hom}_R(S, M)) \cong \mathrm{Ext}^{n+1}_R(U \otimes_S S, M) \cong \mathrm{Ext}^{n+1}_R(U, M).$$

Since $M \in W^nI(R)$, $\mathrm{Ext}^{n+1}_R(U, M) = 0$, and so $\mathrm{Ext}^{n+1}_S(U, \mathrm{Hom}_R(S, M)) = 0$. Hence $\mathrm{Hom}_R(S, M) \in W^nI(S)$.

(2) Let $M \in W^nF(R^{op})$. Then by [1, Proposition 2.6], $M^* \in W^nI(R)$. Hence by (1), $\mathrm{Hom}_R(S, M^*) \in W^nI(S)$. Since by [9, Theorem 2.76], $\mathrm{Hom}_R(S, M^*) \cong (M \otimes_R S)^*$, we deduce that $(M \otimes_R S)^* \in W^nI(S)$, and again by [1, Proposition 2.6], $M \otimes_R S \in W^nF(S^{op})$. □

Definition 3.5. [1, Definition 3.1] The *n-weak injective dimension* of a left module M and *n-weak flat dimension* of a right module N are defined by

n-$\mathrm{wid}_R(M)= \inf\{k : \mathrm{Ext}^{k+1}_R(K_{n-1}, M) = 0, \text{ for any } K_{n-1} \in SSFP(R)\}$,
and
n-$\mathrm{wfd}_R(N)= \inf\{k : \mathrm{Tor}^R_{k+1}(N, K_{n-1}) = 0, \text{ for any } K_{n-1} \in SSFP(R)\}$.

Corollary 3.6. *Consider an almost excellent extension $S \geq R$. Then*

(1) n-$\mathrm{wid}_S(\mathrm{Hom}_R(S, M)) \leq n$-$\mathrm{wid}_R(M)$ *for a left R-module M.*
(2) n-$\mathrm{wfd}_S(M \otimes_R S) \leq n$-$\mathrm{wfd}_R(M)$ *for a right R-module M.*

Proof. (1) Let $K_{n-1} \in SSFP(S)$ with respect to every $U \in S^nFP(S)$. If n-wid$_S(\mathrm{Hom}_R(S,M)) = k$, then $\mathrm{Ext}_S^{k+1}(K_{n-1}, \mathrm{Hom}_R(S,M)) = 0$. By Lemma 3.3, $K_{n-1} \in SSFP(R)$, too. We have:

$$\mathrm{Ext}_R^{k+1}(K_{n-1}, M) = \mathrm{Ext}_R^{k+1}(K_{n-1} \otimes_S S, M) \cong \mathrm{Ext}_S^{k+1}(K_{n-1}, \mathrm{Hom}_R(S,M)) = 0.$$

So $\mathrm{Ext}_R^{k+1}(K_{n-1}, M) = 0$, and hence $k \leq$n-wid$_R(M)$.

(2) Let $K_{n-1} \in SSFP(S)$ with respect to every $U \in S^nFP(S)$. If n-wfd$_S(M \otimes_R S) = k$, then $\mathrm{Tor}_{k+1}^S(M \otimes_R S, K_{n-1}) = 0$. Also,

$$0 = \mathrm{Ext}_S^{k+1}(K_{n-1}, (M \otimes_R S)^*) \cong \mathrm{Ext}_S^{k+1}(K_{n-1}, \mathrm{Hom}_R(S, M^*)) \cong \mathrm{Ext}_R^{k+1}(K_{n-1}, M^*).$$

So, $k \leq$n-wid$_R(M^*)$ if and only if $k \leq$n-wfd$_R(M)$. □

In the following, we give a condition under which the converse of the previous results is true.

Theorem 3.7. *Consider an excellent extension $S \geq R$. Then*

(1) *$M \in W^nI(R)$ if and only if $\mathrm{Hom}_R(S,M) \in W^nI(S)$.*
(2) *$M \in W^nF(R^{op})$ if and only if $M \otimes_R S \in W^nF(S^{op})$.*

Proof. (1) ($\Longrightarrow$) Follows from Proposition 3.4(1).

($\Longleftarrow$) Let $U \in S^nFP(R)$. Then by Lemma 3.2, $S \otimes_R U \in S^nFP(S)$. So, we have

$$0 = \mathrm{Ext}_S^{n+1}(S \otimes_R U, \mathrm{Hom}_R(S,M)) \cong \mathrm{Ext}_R^{n+1}(U, \mathrm{Hom}_R(S,M)).$$

Since S is a finitely generated free R-module, we have

$$0 = \mathrm{Ext}_R^{n+1}(U, \mathrm{Hom}_R(S,M)) \cong \mathrm{Ext}_R^{n+1}(U, \mathrm{Hom}_R(\oplus_{i=1}^m R, M)) \cong \mathrm{Ext}_R^{n+1}(U, \prod_{i=1}^m M).$$

Therefore, $\prod_{i=1}^m M \in W^nI(R)$, and hence by [1, Proposition 2.9], $M \in W^nI(R)$.

(2) ($\Longrightarrow$) is clear by Proposition 3.4(2).

($\Longleftarrow$) If $M \otimes_R S \in W^nF(S^{op})$, then by [1, Proposition 2.6], $(M \otimes_R S)^* \cong \mathrm{Hom}_R(S, M^*) \in W^nI(S)$. So by (1), $M^* \in W^nI(R)$, and hence by [1, Proposition 2.6], $M \in W^nF(R^{op})$. □

Corollary 3.8. *Consider an excellent extension $S \geq R$. Then*

(1) *n-wid$_S(\mathrm{Hom}_R(S,M)) =$$n$-wid$_R(M)$ for a left R-module M.*
(2) *n-wfd$_S(S \otimes_R M) =$$n$-wfd$_R(M)$ for a right R-module M.*

Proof. (1) By Corollary 3.6, n-$\mathrm{wid}_S(\mathrm{Hom}_R(S,M)) \leq n$-$\mathrm{wid}_R(M)$. We prove that n-$\mathrm{wid}_R(M) \leq n$-$\mathrm{wid}_S(\mathrm{Hom}_R(S,M))$. If $U \in S^nFP(R)$, then by Lemma 3.2, $S \otimes_R U \in S^nFP(S)$. So, if $K_{n-1} \in SSFP(R)$ with respect to U, then we deduce that $K_{n-1} \otimes_R S \in SSFP(S)$ with respect to $S \otimes_R U$. Now suppose that n-$\mathrm{wid}_S(\mathrm{Hom}_R(S,M)) = k$, then by [1, Proposition 3.2], $\mathrm{Ext}_S^{k+1}(K_{n-1} \otimes_R S, \mathrm{Hom}_R(S,M)) = 0$. Also, S is a finitely generated free R-module, so we have $0 = \mathrm{Ext}_S^{k+1}(K_{n-1} \otimes_R S, \mathrm{Hom}_R(S,M)) \cong \mathrm{Ext}_R^{k+1}((K_{n-1} \otimes_R S) \otimes_S S, M) \cong \mathrm{Ext}_R^{k+1}(K_{n-1} \otimes_R S, M) \cong \mathrm{Ext}_R^{k+1}(\oplus_1^m K_{n-1}, M) \cong \prod_{i=1}^m \mathrm{Ext}_R^{k+1}(K_{n-1}, M)$. Hence again by [1, Proposition 3.2], we get that n-$\mathrm{wid}_R(M) \leq k$, and so n-$\mathrm{wid}_R(M) \leq n$-$\mathrm{wid}_S(\mathrm{Hom}_R(S,M))$. Similarly, by Corollary 3.6, [1, Proposition 3.3] and Theorem 3.7, (2) holds. □

In the following, we give equivalent conditions for ${}_SM \in W^nI(R)$ and also $M_S \in W^nF(R^{op})$ under an almost excellent extension of rings.

Proposition 3.9. *Consider an almost excellent extension $S \geq R$. Then there are the following equivalent expressions:*

(1) ${}_SM \in W^nI(R)$.
(2) $\mathrm{Hom}_R(S, {}_SM) \in W^nI(S)$.
(3) ${}_SM \in W^nI(S)$.

Proof. (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1) are clear by Proposition 3.4 and Lemma 3.3, respectively.

(2) $\Longrightarrow$ (3) Similar to the proof of [10, Lemma 2.3(2)], ${}_SM$ is isomorphic to a direct summand of $\mathrm{Hom}_R(S,M)$. Hence by [1, Proposition 2.9], $M \in W^nI(S)$. □

Corollary 3.10. *Consider an almost excellent extension $S \geq R$. Then there are the following equivalent expressions:*

(1) $M_S \in W^nF(R^{op})$.
(2) $M_S \otimes_R S \in W^nF(S^{op})$.
(3) $M_S \in W^nF(S^{op})$.

Proof. Trivial by [1, Propositions 2.6 and 2.9] and Proposition 3.9. □

Corollary 3.11. *Consider an almost excellent extension $S \geq R$. Then*

(1) n-wid$_S(M)$ =n-wid$_R(M)$ *for a left S-module M.*
(2) n-wfd$_S(M)$ =n-wfd$_R(M)$ *for a right S-module M.*

Proof. (1) Let n-wid$_R(M) = k$. If $k = 0$, then by Proposition 3.9, $M \in W^nI(R)$ if and only if $M \in W^nI(S)$, so n-wid$_S(M) = 0$. If $k > 0$ and $0 \to M \to E \to L \to 0$ is a short exact sequence of S-modules with injective module E, then by using of [15, Proposition 5.1] and induction hypothesis, we deduce that n-wid$_R(M) = k$ if and only if n-wid$_R(L) = k - 1$ if and only if n-wid$_S(L) = k - 1$ if and only if n-wid$_S(M) = k$.

(2) This proof is similar to proof (1). □

Let $n \geq 0$. Then the *n-super finitely presented dimension* of a ring in [1, Definition 4.9] defined as

$$\text{l.n.sp.gl.idim}(R) = \sup\{\text{pd}(K_{n-1}) \mid \text{for every } K_{n-1} \in SSFP(R)\}.$$

If $n = 0$, then the *0-super finitely presented (super finitely presented) dimension* of a ring defined as l.sp.gl.idim$(R) = \sup\{\text{pd}(L) \mid \text{for every } L \in SFP(R)\}$.

We consider the relation between the n-super finitely presented dimensions of S and R.

Theorem 3.12. *Consider an excellent extension $S \geq R$. Then* l.n.sp.gl.idim(R) = l.n.sp.gl.idim(S).

Proof. Let l.n.sp.gl.idim$(R) = m$. Then

$$\text{l.n.sp.gl.idim}(R) = \sup\{\text{pd}(K_{n-1}) \mid \text{for every } K_{n-1} \in SSFP(R)\} = m.$$

So by [1, Proposition 3.2], n-wid$_R(M) \leq m$ for every $_RM$. Hence for every $_SN$, n-wid$_R(N) \leq m$. Therefore by Corollary 3.11, n-wid$_S(N) \leq m$, and so l.n.sp.gl.idim$(S) \leq$ l.n.sp.gl.idim(R) for any $n \geq 0$. Conversely, suppose that l.n.sp.gl.idim$(S) = m$. Then l.n.sp.gl.idim$(S) = \sup\{\text{pd}(K_{n-1}) \mid$ for every $K_{n-1} \in SSFP(S)\} = m$. So by [1, Proposition 3.2], n-wid$_S(N) \leq m$ for every N. Hence for every $_RM$, n-wid$_S(M \otimes_R S) \leq m$. So by Corollary 3.11, n-wid$_R(M \otimes_R S) \leq m$. Since S is a finitely generated free R-module, we deduce that n-wid$_R(M \otimes_R \oplus_{i=1}^t R) \leq m$. Also, we have n-wid$_R(M) \leq n$-wid$_R(M^t) \leq m$. Hence, it follows that l.n.sp.gl.idim$(R) \leq$ l.n.sp.gl.idim(S) for any $n \geq 0$. □

4. Gorenstein n-weak injective and n-weak flat modules

In the present section, we examine properties of the classes $GW^nI(R)$, $GW^nF(R^{op})$ and also dimensions under change of rings.

Proposition 4.1. *Consider an almost excellent extension $S \geq R$. Then*

(1) *If $M \in GW^nI(R)$, then $\operatorname{Hom}_R(S, M) \in GW^nI(S)$.*
(2) *If $M \in GW^nF(R^{op})$, then $M \otimes_R S \in GW^nF(S^{op})$.*

Proof. (1) Let $M \in GW^nI(R)$. Then there is an exact sequence

$$\mathbf{WI} = \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots,$$

where $X_i, X^i \in W^nI(R)$ and $M = \operatorname{Ker}(X^0 \to X^1)$. By Proposition 3.4(1), $\operatorname{Hom}_R(S, X_i) \in W^nI(S)$ and $\operatorname{Hom}_R(S, X^i) \in W^nI(S)$. Since S_R is projective, there is an exact sequence

$$\operatorname{Hom}_R(S, \mathbf{WI}) = \cdots \longrightarrow \operatorname{Hom}_R(S, X^0) \longrightarrow \operatorname{Hom}_R(S, X^1) \longrightarrow \cdots,$$

where $\operatorname{Hom}_R(S, X_i) \in W^nI(S)$, $\operatorname{Hom}_R(S, X^i) \in W^nI(S)$ and $\operatorname{Hom}_R(S, M) = \operatorname{Ker}(\operatorname{Hom}_R(S, X^0) \to \operatorname{Hom}_R(S, X^1))$.

(2) If $M \in GW^nF(R^{op})$, then by [2, Theorem 3.10], $M^* \in GW^nI(R)$. So by (1), $\operatorname{Hom}_R(S, M^*) \in GW^nI(S)$. Since by [6, Lemma 2.1], $\operatorname{Hom}_R(S, M^*) \cong (M \otimes_R S)^*$, we have $(M \otimes_R S)^* \in GW^nI(S)$, and then by [2, Theorem 3.10], $M \otimes_R S \in GW^nF(S^{op})$. □

Definition 4.2. Let $n \geq 0$. Then:

(1) We say that a left R-module M has *Gorenstein n-weak injective dimension at most m*, denoted by $\text{G}.n\text{-wid}_R(M) \leq m$, if an exact sequence

$$0 \longrightarrow M \longrightarrow GI_0 \longrightarrow GI_1 \longrightarrow GI_2 \longrightarrow \cdots \longrightarrow GI_m \longrightarrow 0,$$

where $GI_i \in GW^nI(R)$ exists, and say $\text{G}.n\text{-wid}_R(M) = m$ if there is not a shorter exact sequence from the modules belonging to the class $GW^nI(R)$.

(2) We say that a right R-module M has *Gorenstein n-weak flat dimension at most m*, denoted by $\text{G}.n\text{-wfd}_R(M) \leq m$, if an exact sequence

$$0 \longrightarrow GF_m \longrightarrow \cdots \longrightarrow GF_2 \longrightarrow GF_1 \longrightarrow GF_0 \longrightarrow M \longrightarrow 0,$$

where $GF_i \in GW^nF(R^{op})$ exists, and say $\text{G}.n\text{-wfd}_R(M) = m$ if there is not a shorter exact sequence from the modules belonging to the class $GW^nF(R^{op})$.

(3) We define the *Gorenstein n-super finitely presented dimension* and *weak Gorenstein n-super finitely presented dimension* of a ring as $\text{l.G.n.sp.gl.idim}(R) = \sup\{\text{G}.n\text{-wid}_R(M) \mid M \text{ is a left } R\text{-module}\}$ and $\text{r.G.n.sp.gl.fdim}(R) = \sup\{\text{G}.n\text{-wfd}_R(M) \mid M \text{ is a right } R\text{-module}\}$, respectively.

Example 4.3. Let K be a field, and let E be a K-vector space with infinite rank. Set $R = K \ltimes E$ the trivial extension of K by E. Then by [16, Example 3.8], it follows that every R-module is 2-FP-injective and 2-FP-flat. Since every 2-FP-injective (resp., 2-FP-flat) R-module is in $W^2I(R)$ (resp., $W^2F(R^{op})$), we obtain that every R-module is in $GW^2I(R)$ (resp., $GW^2F(R^{op})$), and hence $\text{G}.2\text{-wid}_R(M) = \text{G}.2\text{-wfd}_R(M) = 0$.

Example 4.4. Let R be a ring such that for every R-module M, we have $\mathrm{wid}_R(M) \leq 1$ (resp., $\mathrm{wfd}_R(M) \leq 1$), but R is not regular. For example, $R = k[[x]] \oplus S$, i.e., a coherent $(1, 1, 2)$-ring, where $k[[x]]$ is the ring of power series in 1 indeterminate over a field k, and S is a $(0, 1, 2)$-ring, see examples of [8]. Hence by [1, Proposition 3.2], for any R-module M, there is an exact sequence $0 \to M \to E \to L \to 0$, where E and L are in $WI(R)$. We deduce that E and L are in $GWI(R)$ and hence $\text{G-wid}_R(M) \leq 1$. Now suppose that $\text{G-wid}_R(M) = 0$. Then M is in $GWI(R)$. On the other hand, $\text{l.sp.gl.dim}(R) = 1$ by [5, Theorem 3.8]. So by [2, Proposition 3.15], we obtain that M is in $WI(R)$, and so every module in $FP(R)$ is projective. Hence R is regular, a contradiction. Thus, we get that $\text{G-wid}_R(M) = 1$, and then $\text{l.G.sp.gl.idim}(R) = 1$. Similarly, it follows that $\text{r.G.sp.gl.fdim}(R) = 1$.

Proposition 4.1 implies that the following result holds.

Corollary 4.5. *Consider an almost excellent extension $S \geq R$. Then*

(1) $\text{G}.n\text{-wid}_S(\mathrm{Hom}_R(S, M)) \leq \text{G}.n\text{-wid}_R(M)$ *for a left R-module M.*
(2) $\text{G}.n\text{-wfd}_S(M \otimes_R S) \leq \text{G}.n\text{-wfd}_R(M)$ *for a right R-module M.*

In the following, we deduce that, over an excellent extension, the converse of Proposition 4.1 is also true.

Theorem 4.6. *Consider an excellent extension $S \geq R$. Then*

(1) $M \in GW^nI(R)$ *if and only if* $\mathrm{Hom}_R(S, M) \in GW^nI(S)$.
(2) $M \in GW^nF(R^{op})$ *if and only if* $M \otimes_R S \in GW^nF(S^{op})$.

Proof. (1) ($\Longrightarrow$) It is trivial by Proposition 4.1(1).

($\Longleftarrow$) Let $\mathrm{Hom}_R(S, M) \in GW^nI(S)$. So, an exact sequence

$$\mathbf{WI} = \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots,$$

exists, where $X_i, X^i \in W^nI(S)$ and $\mathrm{Hom}_R(S, M) = \mathrm{Ker}(X^0 \to X^1)$. Since S is a finitely generated free R-module, we have:

$$\mathrm{Hom}_R(S, M) = \mathrm{Hom}_R(\bigoplus_{i=1}^{m} R, M) = \prod_{i=1}^{m} M = \mathrm{Ker}(X^0 \to X^1).$$

Hence, $\prod_{i=1}^{m} M \in GW^nI(R)$, and so by [2, Proposition 3.14], $M \in GW^nI(R)$, too.

(2) ($\Longrightarrow$) It is true according to Proposition 4.1(2).

($\Longleftarrow$) If $M \otimes_R S \in GW^nF(S^{op})$, then by [2, Theorem 3.10], $(M \otimes_R S)^* \cong \mathrm{Hom}_R(S, M^*) \in GW^nI(S)$. So by (1), $M^* \in GW^nI(R)$, and again by [2, Theorem 3.10], $M \in GW^nF(R^{op})$. □

Corollary 4.7. *Consider an excellent extension $S \geq R$. Then*

(1) $\text{G}.n\text{-wid}_S(\mathrm{Hom}_R(S, M)) = \text{G}.n\text{-wid}_R(M)$ *for a left R-module M.*
(2) $\text{G}.n\text{-wfd}_S(S \otimes_R M) = \text{G}.n\text{-wfd}_R(M)$ *for a right R-module M.*

Proof. It is clear by Theorem 4.6. □

Proposition 4.8. *Consider an almost excellent extension $S \geq R$. Then there are the following equivalent expressions:*

(1) ${}_SM \in GW^nI(R)$.
(2) $\mathrm{Hom}_R(S, {}_SM) \in GW^nI(S)$.
(3) ${}_SM \in GW^nI(S)$.

Proof. (1) $\Longrightarrow$ (2) Holds by Proposition 4.1.

(3) $\Longrightarrow$ (1) If $M \in GW^nI(S)$, then there is an exact sequence

$$\mathbf{WI} = \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots,$$

where $X_i, X^i \in W^nI(S)$ and ${}_SM = \mathrm{Ker}(X^0 \to X^1)$. By Proposition 3.9, $X_i, X^i \in W^nI(R)$. Hence similar to the proof of Theorem 4.6(1)($\Longleftarrow$), we deduce that $M \in GW^nI(R)$.

(2) $\Longrightarrow$ (3) Similar to the proof of [10, Lemma 2.3(2)], ${}_SM$ is isomorphic to a direct summand of $\mathrm{Hom}_R(S, M)$. Hence by [2, Proposition 3.14], $M \in GW^nI(S)$. □

From [2, Theorem 3.10 and Proposition 3.14] and Proposition 4.8, we have:

Corollary 4.9. *Consider an almost excellent extension $S \geq R$. Then there are the following equivalent expressions:*

(1) $M_S \in GW^nF(R^{op})$.
(2) $M_S \otimes_R S \in GW^nF(S^{op})$.
(3) ${}_SM \in GW^nF(S^{op})$.

From Proposition 4.8 and Corollary 4.9, we have:

Corollary 4.10. *Consider an almost excellent extension $S \geq R$. Then*

(1) $\mathrm{G}.n\text{-wid}_S(M) = \mathrm{G}.n\text{-wid}_R(M)$ *for a left S-module M.*
(2) $\mathrm{G}.n\text{-wfd}_S(M) = \mathrm{G}.n\text{-wfd}_R(M)$ *for a right S-module M.*

In the following, we consider the relation between the Gorenstein n-super finitely presented dimensions of R and S.

Theorem 4.11. *Consider an excellent extension $S \geq R$. Then for any $n \geq 0$*

(1) $\mathrm{l.G.n.sp.gl.idim}(R) = \mathrm{l.G.n.sp.gl.idim}(S)$.
(2) $\mathrm{r.G.n.sp.gl.fdim}(R) = \mathrm{r.G.n.sp.gl.fdim}(S)$.

Proof. (1) Assume that $\mathrm{l.G.n.sp.gl.idim}(R) = m$. Then $\mathrm{G}.n\text{-wid}_R(M) \leq m$ for every ${}_RM$. Thus, $\mathrm{G}.n\text{-wid}_R(N) \leq m$ for any ${}_SN$, too. So, an exact sequence

$$0 \longrightarrow N \longrightarrow GI_0 \longrightarrow GI_1 \longrightarrow GI_2 \longrightarrow \cdots \longrightarrow GI_m \longrightarrow 0$$

exists, where $GI_i \in GW^nI(S)$.

Since S is a projective R-module, by Theorem 4.6(1), the above exact sequence induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(S, N) \longrightarrow \mathrm{Hom}_R(S, GI_0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(S, GI_m) \longrightarrow 0,$$

where $\mathrm{Hom}_R(S, GI_i) \in GW^nI(S)$. So, G.$n$-wid$_S(\mathrm{Hom}_R(S, N)) \leq m$. By hypothesis, we have $S \cong \bigoplus_{i=1}^{t} R$. Hence G.$n$-wid$_S(N^t) \leq m$, and so G.$n$-wid$_S(N) \leq m$. Consequently, l.G.n.sp.gl.idim$(S) \leq$ l.G.n.sp.gl.idim(R).

Now let l.G.n.sp.gl.idim$(S) = m$. Then G.n-wid$_S(N) \leq m$ for every $_SN$. Thus, G.n-wid$_S(\mathrm{Hom}_R(S, M)) \leq m$ for every $_RM$. So, there is a resolution of $\mathrm{Hom}_R(S, M)$ in $GW^nI(S)$:

$$0 \longrightarrow \mathrm{Hom}_R(S, M) \longrightarrow GI_0 \longrightarrow GI_1 \longrightarrow \cdots \longrightarrow GI_m \longrightarrow 0,$$

where by Proposition 4.8, every $GI_i \in GW^nI(R)$, too. Also, by hypothesis we have $S \cong \bigoplus_{i=1}^{t} R$. So $\mathrm{Hom}_R(S, M) = \prod_{i=1}^{t} M$, and then G.$n$-wid$_R(\prod_{i=1}^{t} M) \leq m$. Therefore G.$n$-wid$_R(M) \leq m$ and therefore, l.G.n.sp.gl.idim$(R) \leq$ l.G.n.sp.gl.idim(S).

(2) Suppose that r.G.n.sp.gl.fdim$(R) = t$. Then G.n-wfd$_R(M) \leq t$ for every M_R. Thus, G.n-wfd$_R(N) \leq t$ for every N_S, too. So, there is an exact sequence of the form:

$$0 \longrightarrow GF_t \longrightarrow \cdots \longrightarrow GF_1 \longrightarrow GF_0 \longrightarrow N \longrightarrow 0,$$

where $GF_i \in GW^nF(R^{op})$.

Since S is a flat R-module, by Theorem 4.6(2), the above exact sequence induces an exact sequence

$$0 \longrightarrow GF_t \otimes_R S \longrightarrow \cdots \longrightarrow GF_1 \otimes_R S \longrightarrow GF_0 \otimes_R S \longrightarrow N \otimes_R S \longrightarrow 0,$$

where $GF_i \otimes_R S \in GW^nF(S^{op})$. So, G.$n$-wfd$_S(N \otimes_R S) \leq t$. Hence G.$n$-wfd$_S(N^t) \leq t$, and so G.$n$-wfd$_S(N) \leq t$. Thus, r.G.n.sp.gl.fdim$(S) \leq$ r.G.n.sp.gl.fdim(R).

Now assume that r.G.n.sp.gl.fdim$(S) = t$. Then G.n-wfd$_S(N) \leq t$ for every S-module N. Thus, G.n-wfd$_S(M \otimes_R S) \leq t$ for every M_R. So, there is a resolution of $M \otimes_R S$ in $GW^nF(S^{op})$:

$$0 \longrightarrow GF_t \longrightarrow \cdots \longrightarrow GF_1 \longrightarrow GF_0 \longrightarrow M \otimes_R S \longrightarrow 0,$$

where by Corollary 4.9, every GF_i is in $GW^nF(R^{op})$, too. Also, $M \otimes_R S \cong \bigoplus_{i=1}^{l} M$. So, G.$n$-wfd$_R(\bigoplus_{i=1}^{l} M) \leq t$, and then G.$n$-wfd$_R(M) \leq t$. Therefore, r.G.n.sp.gl.fdim$(R) \leq$ r.G.n.sp.gl.fdim(S). □

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