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TWO-SIDED SGUT-MAJORIZATION AND ITS LINEAR PRESERVERS

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Abstract. Let $\mathbf{M}_{n,m}$ be the set of all n-by-m real matrices, and let $\mathbb{R}^n$ be the set of all n-by-1 real vectors. An n-by-m matrix $R = [r_{ij}]$ is called g-row substochastic if $\sum_{k=1}^{m} r_{ik} \leq 1$ for all i $(1 \leq i \leq n)$. For $x, y \in \mathbb{R}^n$, it is said that x is *sgut-majorized* by y, and we write $x \prec_{sgut} y$ if there exists an n-by-n upper triangular g-row substochastic matrix R such that $x = Ry$.

Define the relation $\sim_{sgut}$ as follows. $x \sim_{sgut} y$ if and only if x is sgut-majorized by y and y is sgut-majorized by x. This paper characterizes all (strong) linear preservers of $\sim_{sgut}$ on $\mathbb{R}^n$.

Keywords: Generalized row substochastic matrix, (strong) Linear preserver, Two-sided sgut-majorization.
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1. Introduction

Over the years, the theory of majorization has been used as a powerful tool in applied and pure mathematics. Majorization is a pre-ordering on vectors by sorting all components in non-increasing order, i.e., for each x, $y \in \mathbb{R}^n$ the vector x is said to be majorized by y $(x \prec y)$, if $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for all $1 \leq k \leq n$ with equality for $k = n$, where $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})$ is the non-increasing rearrangement of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The history of its research goes back to [6] and [12]. The reader can find in-depth information about this concept in [11]. Ando in a basic paper [1] characterized the structure of linear preservers of this relation. In 1991 Dahl generalized the majorization concept to matrices. Ando [2] did a basic investigation on the theory of majorization. In 2005, the authors [5] introduced a new structure of doubly stochastic matrices. Those interested can refer to [3, 4, 7, 8, 10] for more information. Here, we introduce the relation $\sim_{sgut}$ and we obtain all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ (strongly) preserving this relation.

Throughout the article, $\mathcal{RS}_n^{gut}$ denotes the collection of all n-by-n upper triangular g-row substochastic matrices, $\{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathbb{R}^n$, $A(n_1, \ldots, n_l|m_1, \ldots, m_k)$ denotes the submatrix of A obtained from A by deleting rows $n_1, \ldots, n_l$ and columns $m_1, \ldots, m_k$. r_i denotes the sum

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of the entries of the i row of A, $A(n_1, \dots, n_l)$ denotes the abbreviation of $A(n_1, \dots, n_l|n_1, \dots, n_l)$, $\mathbb{N}_k$ denotes the set $\{1, \dots, k\} \subset \mathbb{N}$, A^t denotes the transpose of a given matrix $A \in \mathbf{M}_n$, $[T]$ denotes the matrix representation of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with respect to the standard basis, and $\mathcal{A}(S)$ denotes the set $\{\sum_{i=1}^m \lambda_i a_i \mid m \in \mathbb{N},\ \sum_{i=1}^m \lambda_i \le 1,\ \ \lambda_i \ge 0,\ \ a_i \in S,\ \ \forall i \in \mathbb{N}_m\}$, where $S \subseteq \mathbb{R}^n$.

Let $\mathcal{R}$ be a relation on $\mathcal{V}$, where $\mathcal{V}$ is a linear space of matrices. A linear transformation $T : \mathcal{V} \to \mathcal{V}$ is linearly preserver of $\mathcal{R}$ if $\mathcal{R}(TX, TY)$ whenever $\mathcal{R}(X, Y)$. If T is a linear preserver of $\mathcal{R}$ and $\mathcal{R}(TX, TY)$ implies that $\mathcal{R}(X, Y)$, then T is called a strong linear preserver of $\mathcal{R}$.

A matrix is called g-row substochastic if the sum of the entries of each row should be less than or equal to one. Let x, $y \in \mathbb{R}^n$. We say that x is sgut-majorized by y, written $x \prec_{sgut} y$, if $x = Ry$ for some $R \in \mathcal{RS}_n^{gut}$.

In [9], all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ (strong) preserving sgut-majorization found, as follow.
Although the main results of this paper and [9] are the same, the key techniques in the proofs are different. For example, see the proofs of Theorem 2.6 ([9]) and the following theorem.

Theorem 1.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $[T] = [a_{ij}]$. Then T preserves $\prec_{sgut}$ if and only if one of the following options occurs:*
(a) *$n - 1$ up to the first column of $[T]$ are zero.*
(b) *There exist $t \in \mathbb{N}_{n-1}$ and $1 \le i_1 < \cdots < i_m \le n$ such that $a_{i_1 t}, a_{i_2 (t+1)}, \dots, a_{i_m n}$ are not zero,*

$$[T] = \begin{pmatrix} 0 & * & & & & & \\ & a_{i_1 t} & & * & & & \\ & & \ddots & & & & \\ & & & a_{i_2(t+1)} & & & \\ & & & & \ddots & & \\ & 0 & & & a_{i_{m-1}(n-1)} & & \\ & & & & & \ddots & \\ & & & & & & a_{i_m n} \\ & & & & & & * \end{pmatrix},$$

and one of the following statement happens.
(i) *Define h_m equal to the collection of the total entries of rows $i_{m-1} + 1$ to the end. Then* $\mathrm{card(h_m)} \ge 2$.
(ii) *Define h_1 equal to the collection of the total entries of rows 1 to the $i_1 - 1$ and the row n and h_j equal to the collection of the total entries of rows $i_{j-1} + 1$ to the $i_j - 1$ and the row n for each j $(2 \le j \le m - 1)$. There exists $k \in \mathbb{N}_{m-1}$ such that* $\mathrm{card(h_k)} \ge 2$*, $r_{i_k} = r_{i_k+1} = \cdots = r_n$, and for each $i \ge i_k$, and for each $j \in \mathbb{N}_n$, $a_{ij} \ge 0$ or $a_{ij} \le 0$.*
(iii) *The totals of each row should be equal and have the same signs.*

Theorem 1.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then T strongly preserves $\prec_{sgut}$ if and only if $[T] = \alpha I_n$ for some $\alpha \in \mathbb{R} \setminus \{0\}$.*

In this paper, after introducing the relation $\sim_{sgut}$ we get all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ (strongly) preserving sgut-majorization.

2. Main results

Here, by expressing the relation g-row substochastic matrices we find the structure of (strong) linear preservers of that on $\mathbb{R}^n$.

Definition 2.1. Let $x, y \in \mathbb{R}^n$. Then x two-sided sgut-majorized by y (in symbol $x \sim_{sgut} y$) if $x \prec_{sgut} y \prec_{sgut} x$.

Pay attention to the following proposition for sgut-majorization on $\mathbb{R}^n$.

Proposition 2.2. *Let $x = (x_1, \dots, x_n)^t$, $y = (y_1, \dots, y_n)^t \in \mathbb{R}^n$. Then $x \sim_{sgut} y$ if and only if for all $i \in \mathbb{N}_{n-1}$*

$$x_i \in \mathcal{A}\{y_i, \dots, y_n\},$$

$$y_i \in \mathcal{A}\{x_i, \dots, x_n\},$$

and also

$$x_n = y_n$$

or

$$x_n y_n < 0.$$

To prove the main theorems, we need to state the following results.

Lemma 2.3. *Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\sim_{sgut}$. Assume that $U : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ is the linear transformation with $[U] = [T](1, \dots, k)$. Then U preserves $\sim_{sgut}$ on $\mathbb{R}^{n-k}$.*

Proof. Let $x' = (x_{k+1}, \dots, x_n)^t$, $y' = (y_{k+1}, \dots, y_n)^t \in \mathbb{R}^{n-k}$, and let $x' \sim_{sgut} y'$. Set $x := \sum_{i=k+1}^{n} x_i$ and $y := \sum_{i=k+1}^{n} y_i$, where $x, y \in \mathbb{R}^n$. We see $x \sim_{sgut} y$, and then $Tx \sim_{sgut} Ty$. This follows that $Ux' \sim_{sgut} Uy'$. Therefore, U preserves $\sim_{sgut}$, as desired. □

Lemma 2.4. *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear preserver of $\sim_{sgut}$, then $[T]$ is upper triangular.*

Proof. Suppose $[T] = [a_{ij}]$. By induction on n we move. Let $n \geq 2$ and the assertion has been established for all linear preservers of $\sim_{sgut}$ on $\mathbb{R}^{n-1}$. If $U : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is the linear transformation with $[U] = [T](1)$, Lemma 2.3 ensures that U preserves $\sim_{sgut}$ on $\mathbb{R}^{n-1}$. So $[U]$ is an $n-1$-by-$n-1$ upper triangular matrix, and we should prove $a_{21} = \cdots = a_{n1} = 0$. For this aim, define

$$I = \{2 \leq i \leq n : a_{i1} \neq 0\}.$$

If I is non-empty; put $t = \max\{i : i \in I\}$. This means that $a_{(t+1)1} = a_{(t+2)1} = \cdots = a_{n1} = 0$, and $a_{t1} \neq 0$. Without loss of generality, $a_{t1} = 1$. We reach the following two cases.

Case 1. $a_{t2} \neq 0$; set $x = -a_{t2}e_1 + e_2$, and $y = y_1e_1 + e_2$, where $y_1 \neq -a_{t2}$. We see $x \sim_{sgut} y$, but $Tx \not\sim_{sgut} Ty$, a contradiction.

Case 2. $a_{t2} = 0$; let $x = e_2$, and $y = e_1 + e_2$. We observe that $x \sim_{sgut} y$, and $Tx \not\sim_{sgut} y$, which is a contradiction.

Thus, I is empty, and $a_{21} = \cdots = a_{n1} = 0$, and we observe that $[T]$ is an upper triangular matrix. □

Lemma 2.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $a_{kt} \neq 0$ for some $k, t \in \mathbb{N}_{n-1}$, where $[T] = [a_{ij}]$. Assume that $a_{k+1t} = a_{k+2t} = \cdots = a_{nt} = 0$, and there exists some j $(t+1 \leq j \leq n)$ such that $a_{k+1j} = a_{k+2j} = \cdots = a_{nj} = 0$. Then T does not preserve $\sim_{sgut}$.*

Proof. We can assume without loss of generality that $a_{kt} = 1$ (T preserves $\sim_{sgut}$ if and only if αT preserves $\sim_{sgut}$ for all $\alpha \in \mathbb{R} \setminus \{0\}$). We consider two cases.

Case 1. $t + 1 \leq j < n$; let $x = e_t$ and $y = -a_{kj}e_t + e_j$. We observe that $x \sim_{sgut} y$, and $Tx \not\sim_{sgut} Ty$.

Case 2. $j = n$; consider $x = e_t + e_n$, and $y = e_n$ whenever $a_{kn} = 0$, and $x = e_n$, and $y = -a_{kt}e_t + e_n$ whenever $a_{kn} \neq 0$. We deduce that $x \sim_{sgut} y$, and $Tx \not\sim_{sgut} y$.

Therefore, T does not preserve $\sim_{sgut}$. □

The following theorem defines structure of the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving two-sided sgut-majorization beautifully.

Theorem 2.6. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Assume $[T] = [a_{ij}]$. Then T preserves $\sim_{sgut}$ if and only if one of the following conditions holds.*

(a) $n - 1$ up to the first column of $[T]$ are zero.

(b) There exist $t \in \mathbb{N}_{n-1}$ and $1 \leq i_1 < \cdots < i_m \leq n$ such that
$a_{i_1t}, a_{i_2t+1}, \ldots, a_{i_mn} \neq 0$,

$$[T] = \begin{pmatrix} 0 & * & & & & & \\ & a_{i_1t} & & & * & & \\ & & \ddots & & & & \\ & & & a_{i_2(t+1)} & & & \\ & & & & \ddots & & \\ & 0 & & & a_{i_{m-1}(n-1)} & & \\ & & & & & \ddots & \\ & & & & & & a_{i_mn} \\ & & & & & & * \end{pmatrix},$$

and one of the following statement happens.
(i) $\mathrm{card}(\mathrm{h_m}) \geq 2$.
(ii) *there exists* $k \in \mathbb{N}_{m-1}$ *such that* $\mathrm{card}(\mathrm{h_k}) \geq 2$*, from the rows* i_k *to* i_n *the totals of each row should be equal and have the same signs.*
(iii) *The totals of each row should be equal and have the same signs,*
where consider h_m *equal to the collection of the total entries of rows* $i_{m-1}+1$ *to the end,* h_1 *equal to the collection of the total entries of rows* 1 *to the* $i_1 - 1$ *and the row* n *and* h_j *equal to the collection of the total entries of rows* $i_{j-1}+1$ *to the* $i_j - 1$ *and the row* n *for each* j $(2 \leq j \leq m-1)$.

Proof. If (a) or (b) holds, and $x = (x_1, \ldots, x_n)^t$, $y = (y_1, \ldots, y_n)^t \in \mathbb{R}^n$ with $x \sim_{sgut} y$;

As $x \sim_{sgut} y$, we have $x \prec_{sgut} y \prec_{sgut} x$. Theorem 1.1 ensures that $Tx \prec_{sgut} Ty \prec_{sgut} Tx$, and hence $Tx \sim_{sgut} Ty$, that is, T preserves $\sim_{sgut}$.

Now, if T preserves $\sim_{sgut}$, $[T] = [a_{ij}]$, and (a) does not occurs, we want to prove (b) holds. Let $n \geq 3$, and statement holds for all $n-1$. Lemma 2.4 ensures that $[T]$ is upper triangular. Let $U : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear transformation with $[U] = [T](1)$. By Lemma 2.3, U preserves $\sim_{sgut}$ on $\mathbb{R}^{n-1}$. By applying the induction hypothesis for U, we should consider two steps.

Step 1. If U staisfies (a); Lemma 2.5 states that the first nonzero column of $[T]$ should be its $(n-1)st$ column. If $\mathrm{card}(\mathrm{h_m}) \geq 2$, then (b)-(i) holds. If not; $r_2 = \cdots = r_n$. Without loss of generality, assume that $a_{1n-1} = 1$. We prove $r_1 = r_n$, $a_{1n}, a_{nn} \geq 0$, and $a_{nn} \neq 0$. Lemma 2.5 ensures that $a_{nn} \neq 0$. If $r_1 \neq r_n$; choose $x_{n-1} \in \mathbb{R} \setminus \{1, a_{nn} - a_{1n}\}$, and put $x = x_{n-1}e_{n-1} + e_n$ and $x = (a_{nn} - a_{1n})e_{n-1} + e_n$. We deduce that $x \sim_{sgut} y$, and then $Tx \sim_{sgut} Ty$. This implies that $x_{n-1} + a_{1n} \in \mathcal{A}\{a_{nn}\}$, which would be a contradiction. Hence $r_1 = r_n$. Now, we claim that $a_{nn} > 0$. If $a_{nn} < 0$; set $x = e_n$ and $y = e_{n-1} + e_n$. We have $x \sim_{sgut} y$, and so $Tx \sim_{sgut} Ty$. We conclude that $a_{1n} \in \mathcal{A}\{a_{nn}\}$. There exists $0 \leq \lambda \leq 1$ such that $a_{1n} = \lambda a_{nn}$. As $a_{nn} < 0$, we see $a_{nn} \leq a_{1n}$, a contradiction. Hence $a_{nn} > 0$.
We claim that $a_{1n} \geq 0$. If $1 > a_{nn} + a_{1n}$; choose x_{n-1} such that $1 > x_{n-1} > a_{nn} + a_{1n}$. Set $x = x_{n-1}e_{n-1} - e_n$, and $y = e_{n-1} + e_n$. We observe that $x \sim_{sgut} y$ and then $Tx \sim_{sgut} Ty$. This follows that $x_{n-1} - a_{1n} \in \mathcal{A}\{a_{nn}\}$. Thus, there exists $\lambda \leq 1$ such that $x_{n-1} - a_{1n} = \lambda a_{nn}$. As $a_{nn} > 0$, we have $x_{n-1} - a_{1n} \leq a_{nn}$, and so $x_{n-1} \leq a_{nn} + a_{1n}$, a contradiction. Hence $1 \leq a_{nn} + a_{1n}$. In this case, $1 \leq (1 + a_{1n}) + a_{1n}$, and so $a_{1n} \geq 0$, as desired. This shows that (iii) holds for $[T]$.

Step 2. If S satisfies (b). Let the first nonzero column of $[U]$ be the t^{th} column of $[T]$. We consider two cases.
Case 1. The first nonzero column of $[T]$ is its t^{th} column. So $i_1 > 1$. If $[U]$ is the forms of (iii), and if $r_1 \neq r_n$, then (ii) holds for $[T]$ with $k = 1$. If not; $r_1 = r_n$. So for each i, j $(2 \leq i, j \leq n)$ $a_{ij} \geq 0$, without loss of generality. We should prove $a_{1t}, \ldots, a_{1n} \geq 0$. Define

$$J_1 = \{t \leq j \leq n : a_{1j} \geq 0\},$$

and

$$J_2 = \{t \leq j \leq n : a_{1j} < 0\}.$$

We claim that $J_2 = \emptyset$. If J_2 is nonempty; we know $r_1 \geq 0$. If $J_1 = \emptyset$, then $r_1 < 0$, a contradiction. So J_1 is nonempty. We have two steps.

Step *I*. $a_{1n} < 0$. If $\sum_{j\in J_1} a_{1j} \leq r_1 + \sum_{j\in J_2} a_{1j}$, then $\sum_{j\in J_2} a_{1j} \geq 0$. It is a contradiction. So $\sum_{j\in J_1} a_{1j} > r_1 + \sum_{j\in J_2} a_{1j}$. Choose x_1 such that

$$\sum_{j\in J_1} a_{1j} > x_1 > r_1 + \sum_{j\in J_2} a_{1j}.$$

Set

$$x = x_1 \sum_{j\in J_1} e_j - (\sum_{j\in J_2} e_j)(\sum_{j\in J_1} a_{1j}),$$

and

$$y = (\sum_{j\in J_1} a_{1j})(\sum_{j=t}^{n} e_j).$$

So $x \sim_{sgut} y$, and then $Tx \sim_{sgut} Ty$. This implies that

$$x_1 \sum_{j\in J_1} a_{1j} - (\sum_{j\in J_2} a_{1j})(\sum_{j\in J_1} a_{1j}) \in \mathcal{A}\{(\sum_{j\in J_1} a_{1j})r_1\}.$$

So there exists $\lambda \leq 1$ such that

$$x_1 \sum_{j\in J_1} a_{1j} - (\sum_{j\in J_2} a_{1j})(\sum_{j\in J_1} a_{1j}) = \lambda(\sum_{j\in J_1} a_{1j})r_1.$$

If $\sum_{j\in J_1} a_{1j} = 0$, then $r_1 < 0$, which is a contradiction. If $\sum_{j\in J_1} a_{1j} \neq 0$, we have $x_1 - \sum_{j\in J_2} a_{1j} \leq r_1$, a contradiction.

Step *II*. $a_{1n} \geq 0$. Put $x = \sum_{j\in J_1} e_j$ and $y = \sum_{j=t}^{n} e_j$. We see $x \sim_{sgut} y$, and then $Tx \sim_{sgut} Ty$. This shows that $\sum_{j\in J_1} a_{1j} \in \mathcal{A}\{r_1\}$. So $\sum_{j\in J_1} a_{1j} \leq r_1$, and hence

$$\sum_{j\in J_1} a_{1j} \leq \sum_{j\in J_1} a_{1j} + \sum_{j\in J_2} a_{1j}.$$

That is, $0 \leq \sum_{j\in J_2} a_{1j}$. It is a contradiction.
Thus, $J_2 = \emptyset$, and $a_{1t}, a_{1t+1}, \ldots, a_{1n} \geq 0$. We observe that (iii) holds for $[T]$.

Case 2. The first nonzero column of $[T]$ is not its t^{th} column. Lemma 2.5 states that the first nonzero column of $[T]$ is its $(t-1)$st column. It is proven in a similar way. □

We need the following lemmas in the rest of this paper.

Lemma 2.7. *Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *be an invertible linear preserver of* $\sim_{sgut}$*, and let* $[T] = [a_{ij}]$*. Then* $[T]$ *is upper triangular,* $\prod_{i=1}^{n} a_{ii} \neq 0$*,* $r_1 = r_2 = \cdots = r_n$ *and for each* $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ *or* $a_{ij} \leq 0$.

Proof. Since T preserves $\sim_{sgut}$, we see $[T]$ is an upper triangular matrix, by Lemma 2.4. On the other hand, as $[T]$ is upper triangular and invertible, we deduce that $\prod_{i=1}^{n} a_{ii} \neq 0$. Now, Theorem 2.6 ensures that $r_1 = r_2 = \cdots = r_n$ and for each $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ or $a_{ij} \leq 0$. □

Lemma 2.8. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation that strongly preserves $\sim_{sgut}$. Then T is invertible.*

Proof. If $x \in \mathbb{R}^n$ and $Tx = 0$; Since T strongly preserves $\sim_{sgut}$, we have $x \sim_{sgut} 0$. So $x = 0$, and the proof is over. □

In the last theorem of this paper, we obtain the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ which strongly preserves two-sided sgut-majorization.

Theorem 2.9. *A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ strongly preserves $\sim_{sgut}$ if and only if $[T]$ is a real non-zero multiple of the identity matrix.*

Proof. We only have to prove if T strongly preserves $\sim_{sgut}$, then $[T]$ is a real non-zero multiple of the identity matrix. Let T strongly preserve $\sim_{sgut}$. This follows that T preserves $\sim_{sgut}$, and T is invertible. Then by Lemma 2.7 $[T]$ is upper triangular, $\prod_{i=1}^{n} a_{ii} \neq 0$, $r_1 = r_2 = \cdots = r_n$, and for each $i, j \in \mathbb{N}_n$ $a_{ij} \geq 0$ or $a_{ij} \leq 0$. By induction on n, we prove the statement. Let $n \geq 2$, and the statement has been proved for all strong linear preservers of $\sim_{sgut}$ on $\mathbb{R}^{n-1}$. Let $U : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the linear transformation with $[U] = [T](1)$. Lemma 2.3 ensures that U preserves $\sim_{sgut}$ on $\mathbb{R}^{n-1}$. We claim that U strongly preserves $\sim_{sgut}$ on $\mathbb{R}^{n-1}$. Let $x' = (x_2, \ldots, x_n)^t$, $y' = (y_2, \ldots, y_n)^t \in \mathbb{R}^{n-1}$, and let $Ux' \sim_{sgut} Uy'$. Set $x = (0, x')^t$ and $y = (0, y')^t \in \mathbb{R}^n$. We see

$$Tx = (\sum_{i=2}^{n} a_{1i} x_i, Ux')^t, \quad Ty = (\sum_{i=2}^{n} a_{1i} y_i, Uy')^t.$$

For proving $Tx \sim_{sgut} Ty$, we should prove

$$(Tx)_1 \in \mathcal{A}\{(Ty)_i\}_{i=1}^{n}, \quad (Ty)_1 \in \mathcal{A}\{(Tx)_i\}_{i=1}^{n}.$$

If $(Ty)_1 = \cdots = (Ty)_n$; we obtain $y_2 = \cdots = y_n$. As $(Ty)_1 = (Ty)_n$, we have $\sum_{i=2}^{n} a_{1i} y_i = a_{nn} y_n$. We know $y_2 = \cdots = y_n$, so $(\sum_{i=2}^{n} a_{1i}) y_n = a_{nn} y_n$. If $y_n \neq 0$, then $\sum_{i=2}^{n} a_{1i} = a_{nn}$. This implies that $a_{11} = 0$, a contradiction. So $y_2 = \cdots = y_n = 0$, and $y' = 0$. This means that $Sy' = 0$, and we deduce that $Sx' = 0$, because $Sx' \sim_{sgut} Sy'$. $(Sx')_n = 0$ shows that $x_n = 0$. Similarly, we prove that $x' = 0$. So $(Tx)_1 = (Ty)_1 = 0$, and we conclude that $(Tx)_1 \in \mathcal{A}\{(Ty)_i\}_{i=1}^{n}$ and $(Ty)_1 \in \mathcal{A}\{(Tx)_i\}_{i=1}^{n}$.
We saw if the vector Ty is a multiple of e, then $x = y = 0$. Similarly, the same thing is proved for Tx, and so $(Tx)_1 \in \mathcal{A}\{(Ty)_i\}_{i=1}^{n}$ and $(Ty)_1 \in \mathcal{A}\{(Tx)_i\}_{i=1}^{n}$. Now, if card $\{(Tx)_i\}_{i=1}^{n} \geq 2$, and if card $\{(Ty)_i\}_{i=1}^{n} \geq 2$, clearly, $(Tx)_1 \in \mathcal{A}\{(Ty)_i\}_{i=1}^{n}$ and $(Ty)_1 \in \mathcal{A}\{(Tx)_i\}_{i=1}^{n}$.
Thus, $Tx \sim_{sgut} Ty$. Since T strongly preserves $\sim_{sgut}$, we deduce that $x \sim_{sgut} y$. This follows that $x' \sim_{sgut} y'$. Hence U strongly preserves $\sim_{sgut}$ on $\mathbb{R}^{n-1}$.

The induction hypothesis ensures that $[U]$ is a real non-zero multiple of the identity matrix. If we prove that $a_{12} = \cdots = a_{1n} = 0$, as $r_1 = \cdots = r_n$, we conclude that $[T]$ is a real non-zero multiple of the identity matrix.

We obtain

$$[T^{-1}] = \begin{pmatrix} \frac{1}{a_{11}} & \frac{-a_{12}}{a_{11}\alpha} & \frac{-a_{13}}{a_{11}\alpha} & \dots & 0 & 0 & \frac{-a_{1n}}{a_{11}\alpha} \\ 0 & \frac{1}{\alpha} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{\alpha} \end{pmatrix}.$$

We see T^{-1} is a linear preserver of $\sim_{sgut}$, because T strongly preserves $\sim_{sgut}$. Theorem 2.6 ensures that all entries of $[T^{-1}]$ have the same sign. As all entries of $[T]$ have the same sign too, it shows that $a_{12} = \cdots = a_{1n} = 0$. □

References

[1] T. Ando, Majorization, doubly stochastic matrices, and comparision of eigenvalues, *Linear Algebra Appl.*, 118 (1989), pp. 163-248.

[2] T. Ando, Majorization and inequalities in matrix theory, *Linear Algebra Appl.*, 199 (1994), pp. 17-67.

[3] A. Armandnejad and Z. Gashool, Strong linear preservers of g-tridiagonal majorization on Rn, *Electronic. J. Linear Algebra*, 23 (2012), pp. 115-121.

[4] A. Armandnejad and A. Ilkhanizadeh Manesh, Gut-majorization and its linear preservers, *Electronic. J. Linear Algebra*, 23 (2012), pp. 646-654.

[5] H. Chiang and C. K. Li, Generalized doubly stochastic matrices and linear preservers, *Linear and Multilinear Algebra*, 53 (2005), pp. 1-11.

[6] G.H. Hardy, J.E. Littlewood, and G. Polya, Some simple inequalities satisfed by convex functions., *Messenger of Mathematics* , 58 (1929), pp. 145-152.

[7] A. M. Hasani and M. Radjabalipour, The structure of linear operators strongly preserving majorizations of matrices, *Electron. J. Linear Algebra*, 15 (2006), pp. 260-268.

[8] A. M. Hasani and M. Radjabalipour, On linear preservers of (right) matrix majorization, *Linear Algebra Appl,* 423 (2007), pp. 255-261.

[9] A. Ilkhanizadeh Manesh, On linear preservers of sgut-majorization on $\mathbf{M}_{n,m}$, *Bull. Iranian Math. Soc.,* 42 (2016), pp. 470-481.

[10] A. Ilkhanizadeh Manesh, Right gut-Majorization on $\mathbf{M}_{n,m}$, *Electron. J. Linear Algebra,* 31 (2016), pp. 13-26.

[11] A. W. Marshall, I. Olkin, and B. C. Arnold, Inequalities: Theory of majorization and its applications, *Springer, New York,* 2011.

[12] I. Schur, Uber enie klasse von mittelbildungen mit anwendungen auf die determinantentheorie, *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, 22 (1923), pp. 9-20.

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